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Curvature estimates for constant mean curvature surfaces

William H. Meeks III* Giuseppe Tinaglia†

Abstract

We derive extrinsic curvature estimates for compact disks embedded in \mathbb{R}^3 with nonzero constant mean curvature.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

Key words and phrases: Minimal surface, constant mean curvature, curvature estimates.

1 Introduction.

For clarity of exposition, we will call an oriented surface M immersed in \mathbb{R}^3 an H -surface if it is *embedded, connected* and it has *positive constant mean curvature* H . We will call an H -surface an H -disk if the H -surface is homeomorphic to a closed disk in the Euclidean plane.

The existence of curvature estimates for compact disks embedded in \mathbb{R}^3 with positive constant mean curvature given below is the main result of this manuscript.

Theorem 1.1 (Extrinsic Curvature Estimates). *Given $\delta, \mathcal{H} > 0$, there exists a constant $K_0(\delta, \mathcal{H})$ such that for any H -disk \mathcal{D} with $H \geq \mathcal{H}$,*

$$\sup_{\{p \in \mathcal{D} \mid d_{\mathbb{R}^3}(p, \partial \mathcal{D}) \geq \delta\}} |A_{\mathcal{D}}| \leq K_0(\delta, \mathcal{H}).$$

We wish to emphasize to the reader that the curvature estimates for embedded constant mean curvature disks given in Theorem 1.1 depend only on the *lower* positive bound \mathcal{H} for their mean curvature. Previous important examples of curvature

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estimates for constant mean curvature surfaces, assuming certain geometric conditions, can be found in the literature; see for instance [2, 3, 4, 9, 10, 14, 27, 28, 29, 30, 34, 35].

We now give a brief outline of our approach to proving Theorem 1.1. The proof of this theorem is by contradiction and relies on an accurate geometric description of a 1-disk near interior points where the norm of the second fundamental form becomes arbitrarily large. This geometric description is inspired by the pioneering work of Colding and Minicozzi in the minimal case [6, 7, 8, 9]; however in the constant positive mean curvature setting this description leads to curvature estimates. Rescalings of a helicoid give rise to a sequence of embedded minimal disks with arbitrarily large norms of their second fundamental forms at points that can be arbitrarily far from their boundary curves; therefore in the minimal setting, curvature estimates do not hold.

Finally, the curvature estimates in Theorem 1.1 are key to proving several fundamental results about the geometry of H -surfaces, see [24, 20, 21, 22]. In turn, these results are used in [19] to generalize the extrinsic curvature estimates to intrinsic ones.

The theory developed in this manuscript also provides key tools for understanding the geometry of H -disks in a Riemannian three-manifold, especially in the case that the manifold is locally homogeneous. These generalizations and applications of the work presented here will appear in our forthcoming paper [18].

2 An Extrinsic Curvature Estimate for certain planar domains.

First, we fix some notations that we use throughout the paper.

- For $r > 0$ and $p \in \mathbb{R}^3$, $\mathbb{B}(p, r) := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |p - x| < r\}$ and $\mathbb{B}(r) := \mathbb{B}(\vec{0}, r)$.
- For $r > 0$ and $p \in \Sigma$, a surface in \mathbb{R}^3 , $B_\Sigma(p, r)$ denotes the open intrinsic ball in Σ of radius r .
- For positive numbers r, h and t ,

$$C(r, h, t) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1 - t)^2 + x_2^2 \leq r^2, |x_3| \leq h\},$$

which is the solid closed vertical cylinder of radius r , height $2h$ and centered at the point $(t, 0, 0)$:

$$C(r, h) := C(r, h, 0).$$

- For positive numbers $r_1 > r_2 > 0$, we let

$$A(r_1, r_2) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid r_2 \leq \sqrt{x_1^2 + x_2^2} \leq r_1, x_3 = 0\},$$

which is the closed annulus in the plane $\{x_3 = 0\}$, centered at the origin with outer radius r_1 and inner radius r_2 .

- For $R > 0$ and $p \in \mathbb{R}^3$, $C_R(p)$ denotes the closed infinite solid vertical cylinder centered at p of radius R and $C_R := C_R(\vec{0})$.
- We let $\Pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $\Pi(x_1, x_2, x_3) = (x_1, x_2)$ denote the orthogonal projection to the (x_1, x_2) -plane.
- We will call a genus-zero surface with boundary a *planar domain*.

The first step in proving the extrinsic curvature estimates for H -disks is to prove extrinsic curvature estimates, Theorem 2.2 below, for certain compact H -surfaces that are planar domains.

Before stating Theorem 2.2, we describe the notion of the flux of a 1-cycle in an H -surface; see for instance [14, 15, 31] for further discussion of this invariant.

Definition 2.1 ([31]). Let γ be a piecewise-smooth 1-cycle in an H -surface M . The flux of γ is $\int_\gamma (H\gamma + \xi) \times \dot{\gamma}$, where ξ is the unit normal to M along γ and γ is parameterized by arc length.

The flux is a homological invariant and we say that M has *zero flux* if the flux of any 1-cycle in M is zero; in particular, since the first homology group of a disk is zero, the flux of an H -disk is zero.

Theorem 2.2. *Given $\varepsilon > 0$, $m \in \mathbb{N}$ and $H \in (0, \frac{1}{2\varepsilon})$, there exists a constant $K(m, \varepsilon, H)$ such that the following holds. Let $M \subset \mathbb{B}(\varepsilon)$ be a compact, connected H -surface of genus zero with at most m boundary components, $\vec{0} \in M$, $\partial M \subset \partial\mathbb{B}(\varepsilon)$ and M has zero flux. Then:*

$$|A_M|(\vec{0}) \leq K(m, \varepsilon, H).$$

Remark 2.3. In Proposition 6.1 we prove that given an H -disk Σ such that $\partial\Sigma \subset \partial\mathbb{B}(\varepsilon)$ with $H \in (0, \frac{1}{2\varepsilon})$, then the number of boundary components of a connected component of $\Sigma \cap \mathbb{B}(\varepsilon)$ is bounded from above by some natural number N_0 that is independent of Σ . Therefore, as the flux of Σ is zero, then Theorem 2.2 together with Proposition 6.1 gives an extrinsic curvature estimates for H -disks.

Namely, suppose $\Sigma \subset \mathbb{R}^3$ is an H -disk, $\vec{0} \in \Sigma$, $\partial\Sigma \subset \partial\mathbb{B}(\varepsilon)$ and $H \in (0, \frac{1}{2\varepsilon})$. Then, by Theorem 2.2, there exists a $K(\varepsilon, H) = K(N_0, \varepsilon, H) > 0$ such that

$$|A_\Sigma|(\vec{0}) \leq K(\varepsilon, H).$$

We first introduce the notion of multi-valued graph, see [7] for further discussion and Figure 1. Intuitively, an N -valued graph is a simply-connected embedded

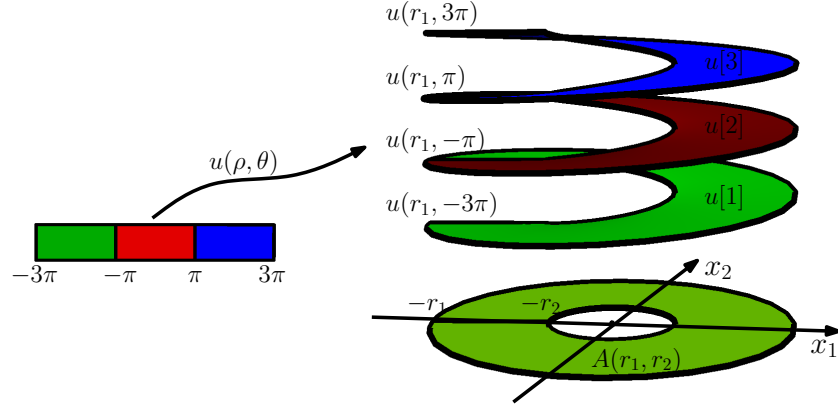


Figure 1: A right-handed 3-valued graph.

surface covering an annulus such that over a neighborhood of each point of the annulus, the surface consists of N graphs. The stereotypical infinite multi-valued graph is half of a helicoid, i.e., half of an infinite double-spiral staircase.

Definition 2.4 (Multi-valued graph). Let \mathcal{P} denote the universal cover of the punctured (x_1, x_2) -plane, $\{(x_1, x_2, 0) \mid (x_1, x_2) \neq (0, 0)\}$, with global coordinates (ρ, θ) , so that $x_1 + ix_2 = \rho e^{i\theta}$.

1. An N -valued graph over the annulus $A(r_1, r_2)$ is a single valued graph $u(\rho, \theta)$ over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1, |\theta| \leq N\pi\} \subset \mathcal{P}$, if N is odd, or over $\{(\rho, \theta) \mid r_2 \leq \rho \leq r_1, (-N+1)\pi \leq \theta \leq \pi(N+1)\} \subset \mathcal{P}$, if N is even.
2. An N -valued graph $u(\rho, \theta)$ over the annulus $A(r_1, r_2)$ is called *righthanded* [*lefthanded*] if whenever it makes sense, $u(\rho, \theta) < u(\rho, \theta + 2\pi)$ [$u(\rho, \theta) > u(\rho, \theta + 2\pi)$]
3. The set $\{(r_2 \cos \theta, r_2 \sin \theta, u(r_2, \theta)) \mid \theta \in [-N\pi, N\pi]\}$ when N is odd (or $\{(r_2 \cos \theta, r_2 \sin \theta, u(r_2, \theta)) \mid \theta \in [(-N+1)\pi, (N+1)\pi]\}$ when N is even) is the *inner boundary* of the N -valued graph.

Note that the boundary of an N -valued graph consists of four connected smooth arcs. They are a spiral on ∂C_{r_2} which is the inner boundary, a similar spiral on

∂C_{r_1} and two arcs γ^\pm connecting the top and bottom pairs of endpoints of these spirals. These latter arcs are of the form

$$\gamma^\pm = \{(t, 0, \phi^\pm(t)) \mid t \in [-r_1, -r_2]\},$$

where $\phi^+(t) = u(-t, N\pi)$ and $\phi^-(t) = u(-t, -N\pi)$.

For simplicity, in the next Definitions 2.5 and 2.6, we assume N is odd and that the N -valued graph is righthanded; the analogous definitions when N is even or the graph is lefthanded are left to the reader. *When we encounter N -valued graphs in the proof of Theorem 2.2, we will also assume, without loss of generality, that N is odd and the N -valued graph is righthanded.*

Definition 2.5. We call the set

$$u[k] := \{(\rho, \theta, u(\rho, \theta)) \mid r_2 \leq \rho \leq r_1, (-N + 2k - 2)\pi \leq \theta \leq (-N + 2k)\pi\},$$

where $k = 1, \dots, N$, the k -th sheet of the N -valued graph and $u_{\text{mid}} := u[\lfloor N/2 \rfloor + 1]$ is its middle sheet; here, $\lfloor N/2 \rfloor$ denotes the integer part of $N/2$.

Definition 2.6. Given an N -valued graph u , $N > 1$, over the annulus $A(r_1, r_2)$ we let $W[u]$ denote the open solid region trapped between the sheets of u . Namely, $W[u]$ is the connected, open, bounded solid region of \mathbb{R}^3 whose boundary consists of the N -valued graph u together with the following union of vertical segments: the set of vertical segments whose end points are $(r_2 \cos(\theta), r_2 \sin(\theta), u(r_2, \theta))$, $(r_2 \cos(\theta + 2\pi), r_2 \sin(\theta + 2\pi), u(r_2, \theta + 2\pi))$ with $\theta \in [-N\pi, (N-2)\pi]$, the set of vertical segments whose end points are $(r_1 \cos(\theta), r_1 \sin(\theta), u(r_1, \theta))$, $(r_1 \cos(\theta + 2\pi), r_1 \sin(\theta + 2\pi), u(r_1, \theta + 2\pi))$ with $\theta \in [-N\pi, (N-2)\pi]$, the set of vertical segments whose end points are $(-\rho, 0, u(\rho, (N-2)\pi))$, $(-\rho, 0, u(\rho, N\pi))$ with $\rho \in [r_2, r_1]$ and the set of vertical segments whose end points are $(-\rho, 0, u(\rho, -N\pi))$, $(-\rho, 0, u(\rho, -(N-2)\pi))$ with $\rho \in [r_2, r_1]$. We can parameterize the set $W[u]$ in a natural way by using coordinates (ρ, θ, x_3) with (ρ, θ, x_3) in an open subset of $(r_2, r_1) \times (-N\pi, (N-2)\pi) \times \mathbb{R}$.

Theorem 2.2 follows in a fairly straightforward way after we prove the following detailed geometric description of a planar domain with constant mean curvature, zero flux and large norm of the second fundamental form at the origin.

Theorem 2.7. *Given $\varepsilon, \tau > 0$, $\bar{\varepsilon} \in (0, \varepsilon/4)$ and $m \in \mathbb{N}$, there exist constants $\Omega_\tau = \Omega(\tau, m) > 0$, $\omega_\tau = \omega(\tau, m) > 0$ and $G_\tau = G(\bar{\varepsilon}, \varepsilon, \tau, m) > 0$ such that if M is a connected compact H -planar domain with zero flux, $H \in (0, \frac{1}{2\varepsilon})$, $M \subset \bar{\mathbb{B}}(\varepsilon)$, $\partial M \subset \partial \bar{\mathbb{B}}(\varepsilon)$ and consists of at most m components, $\vec{0} \in M$ and $|A_M|(\vec{0}) > \frac{1}{\eta} G_\tau$, for $\eta \in (0, 1]$, then for any $p \in \bar{\mathbb{B}}(\eta\bar{\varepsilon})$ that is a maximum of the function $|A_M|(\cdot)(\eta\bar{\varepsilon} - |\cdot|)$, after translating M by $-p$, the following geometric description of M holds.*

- On the scale of the norm of the second fundamental form, M looks like one or two helicoid nearby the origin and, after a rotation that turns these helicoids into vertical helicoids, M contains a 3-valued graph u over $A(\varepsilon/\Omega_\tau, \frac{\omega_\tau}{|A_M|(\vec{0})})$ with norm of the gradient less than τ .
- The intersection $W[u] \cap [M - \text{graph}(u)]$ contains an oppositely oriented 2-valued graph \tilde{u} with norm of the gradient less than τ and $\mathbb{B}(10\frac{\omega_\tau}{|A_M|(\vec{0})}) \cap M$ includes a disk D containing the interior boundaries of $\text{graph}(u)$ and $\text{graph}(\tilde{u})$.
- If near the origin M looks like one helicoid, then the previous description is accurate, namely $W[u] \cap [M - \text{graph}(u)]$ consists of an oppositely oriented 2-valued graph \tilde{u} with norm of the gradient less than τ and $\mathbb{B}(10\frac{\omega_\tau}{|A_M|(\vec{0})}) \cap M$ consists of a disk D containing the interior boundaries of $\text{graph}(u)$ and $\text{graph}(\tilde{u})$.
- If near the origin M looks like two disjoint helicoids, then $W[u] \cap [M - \text{graph}(u) \cup \text{graph}(\tilde{u})]$ consists of a pair of oppositely oriented 2-valued graphs u_1 and \tilde{u}_1 with norm of the gradient less than τ and $\mathbb{B}(10\frac{\omega_\tau}{|A_M|(\vec{0})}) \cap [M - D]$ consists of a disk containing the inner boundaries of $\text{graph}(u_1)$ and $\text{graph}(\tilde{u}_1)$.
- Finally, given $j \in \mathbb{N}$ if we let the constant G_τ depend on j as well, then M contains j disjoint 3-valued graphs and the description in the previous paragraph holds for each of them.

Using Theorem 2.7, the curvature estimates in Theorem 2.2 depend on the nonzero value of the constant mean curvature and are not true for minimal surfaces.

Proof of Theorem 2.2. Arguing by contradiction, suppose that the theorem fails. In this case, for some $\varepsilon \in (0, \frac{1}{2})$, there exists a sequence M_n of H -surfaces satisfying the hypotheses of the theorem and $|A_{M_n}|(\vec{0}) > n$. After replacing M_n with a subsequence and applying a small translation, that we shall still call M_n , Theorem 2.7 show that after composing by a fixed rotation, given any $k \in \mathbb{N}$, there exists an $n(k) \in \mathbb{N}$, such that for $n > n(k)$, there exist $2k$ pairwise disjoint 3-valued graphs $G_{n,1}^{down}, G_{n,1}^{up}, \dots, G_{n,k}^{down}, G_{n,k}^{up}$ in M_n on a fixed horizontal scale, i.e., they are all 3-valued graphs over a fixed annulus A in the (x_1, x_2) -plane, $G_{n,j}^{down} \cap W[G_{n,j}^{up}] \neq \emptyset$ for $j \in \{1, \dots, k\}$, and the gradients of the graphing functions are bounded in norm by 1; here the superscripts “up” and “down” refer to the pointing directions of the unit normals to the graphs.

To obtain a contradiction, note that as the number $2k$ of these pairwise disjoint graphs goes to infinity, there exists a sequence $\{G_{n,j(n)}^{up}, G_{n,j(n)}^{down}\}$ of associated pairs of oppositely oriented 3-valued graphs that collapses smoothly to an annulus of constant mean curvature H that is a graph over A and whose nonzero mean curvature vector points upward and downward at the same time, which is impossible. This contradiction proves that the norm of the second fundamental form of M at the origin must have a uniform bound. \square

We now explain our approach to prove Theorem 2.7. Fix $\varepsilon, \tau > 0$, $\bar{\varepsilon} \in (0, \varepsilon/4)$, $H \in (0, \frac{1}{2\varepsilon})$, $m \in \mathbb{N}$ and $\eta \in (0, 1]$. Let $M_n \subset \overline{\mathbb{B}}(\varepsilon)$, $n \in \mathbb{N}$, be a sequence of H -planar domains such that $\vec{0} \in M_n$, $|A_{M_n}|(\vec{0}) > \frac{n}{\eta}$, and $\partial M_n \subset \partial \mathbb{B}(\varepsilon)$ and consists of at most m components. Since H is fixed, by scaling and after reindexing the elements of the sequence, we can assume that $\varepsilon < \frac{1}{2}$, $H = 1$ and $|A_{M_n}|(\vec{0}) > \frac{n}{\eta}$. We prove that after passing to a subsequence the following statements hold.

1. In Section 3, we show that M_n is closely approximated by one (or two) vertical helicoid on the scale of the norm of the second fundamental form of M_n nearby a point $p \in \overline{\mathbb{B}}(\eta\bar{\varepsilon})$ that is a maximum of the function $|A_M|(\cdot)(\eta\bar{\varepsilon} - |\cdot|)$.
2. In Section 4, we prove that there exists a sequence of embedded stable minimal disks $E(n)$ on the mean convex side of M_n , where $E(n)$ contains a multi-valued graph E_n^g that starts near the origin and extends horizontally on a scale proportional to ε .
3. In Section 5, we use the existence of the minimal multi-valued graph E_n^g to prove that M_n contains a pair G_n^{up}, G_n^{down} of oppositely oriented 3-valued graphs with norms of the gradients bounded by 1 that start near the origin and extend horizontally on a scale proportional to ε , and satisfying $\overline{W}[G_n^{up}] \cap G_n^{down}$ is a 2-valued graph; here, $W[G^{up}]$ is $W[u]$, where u is defined to be the function whose 3-valued graph is G^{up} and \overline{W} denotes the closure.

3 Local picture near a point of large curvature.

In this section we describe the geometry of M_n nearby the origin. Roughly speaking, M_n contains a pair of oppositely oriented multi-valued graphs like in a helicoid (on the scale of the norm of the second fundamental form of M_n at this point.). In the case of embedded minimal disks such a description was given by Colding and Minicozzi in [7]; see also [32, 33] for related results. By rescaling arguments this description can be improved upon once one knows that the helicoid is the unique complete, embedded, non-flat minimal surface in \mathbb{R}^3 as explained below; see [17]

and also [1] for proofs of the uniqueness of the helicoid which are based in part on results in [6, 7, 8, 9, 10].

Let $p_n \in M_n \cap \overline{\mathbb{B}}(\eta\bar{\varepsilon})$ be a maximum for the function

$$f_n: M_n \cap \overline{\mathbb{B}}(\eta\bar{\varepsilon}) \rightarrow [0, \infty), \quad f_n(\cdot) = |A_{M_n}|(\cdot)(\eta\bar{\varepsilon} - |\cdot|);$$

note that the points p_n lie in the interior $\mathbb{B}(\eta\bar{\varepsilon})$ of $\overline{\mathbb{B}}(\eta\bar{\varepsilon})$. We refer to such a p_n as a point of *almost-maximal curvature*.

By a standard compactness argument (see for instance the proof and statement of Theorem 1.1 in [16] or Lemma 5.5 in [17]), given a sequence $p_n \in M_n \cap \mathbb{B}(\eta\bar{\varepsilon})$ of points of almost-maximal curvature, there exist positive numbers δ_n , with $\delta_n \rightarrow 0$, and a subsequence, that we still call M_n , such that the possibly disconnected surfaces $\widehat{M}_n = M_n \cap \mathbb{B}(p_n, \delta_n)$, satisfy:

1. $\lim_{n \rightarrow \infty} \delta_n \cdot |A_{M_n}|(p_n) = \infty$.
2. $\sup_{p \in \widehat{M}_n} |A_{\widehat{M}_n}|(p) \leq (1 + \frac{1}{n}) \cdot |A_{M_n}|(p_n)$.
3. The sequence of translated and rescaled surfaces

$$\Sigma_n = \frac{1}{\sqrt{2}} |A_{M_n}|(p_n) \cdot (\widehat{M}_n - p_n)$$

converges smoothly with multiplicity one or two on compact subsets of \mathbb{R}^3 to a connected, properly embedded, non-flat minimal surface Σ_∞ with bounded norm of the second fundamental form. More precisely,

$$\sup_{\Sigma_\infty} |A_{\Sigma_\infty}| \leq |A_{\Sigma_\infty}|(\vec{0}) = \sqrt{2}.$$

4. In the case that the convergence has multiplicity two, then the mean curvature vectors of the portions of the two surfaces limiting to Σ_∞ point away from the collapsing region between them; recall that the mean curvature vector is $H\xi$ where ξ is the unit normal.
5. Given any smooth loop α in Σ_∞ , for each n sufficiently large, α has a normal lift $\alpha_n \subset \Sigma_n$ such that the lifted loops converge smoothly with multiplicity one to α as $n \rightarrow \infty$; in the case the convergence has multiplicity two, there are exactly two such pairwise disjoint normal lifts of α_n to Σ_n .

We now give some details on obtaining the above description. Regarding the convergence of the surfaces Σ_n to Σ_∞ , the fact that a subsequence of the rescaled surfaces Σ_n converges smoothly to a connected, properly embedded, non-flat minimal surface Σ_∞ with bounded norm of the second fundamental form can be seen

as follows. A standard compactness argument shows that a subsequence of the surfaces Σ_n converges to a non-flat minimal lamination of \mathbb{R}^3 with leaves having uniformly bounded norms of their second fundamental forms; see for instance the proof of Lemma 5.5 in [17] for this type of compactness argument. Hence by Theorem 1.6 in [17], the minimal lamination obtained in the limit consists of a connected, properly embedded, non-flat minimal surface Σ_∞ . Assume, after replacing by such subsequence, that Σ_n converges to Σ_∞ . The convergence of the Σ_n to Σ_∞ is with multiplicity at most two because otherwise a higher multiplicity of convergence would allow for the construction of a positive Jacobi function on Σ_∞ and so Σ_∞ would be a complete stable minimal surface that is a plane [12, 13, 26], which is false; hence the multiplicity of convergence is at most two. The construction of this positive Jacobi function follows the construction of a similar positive Jacobi function on a limit minimal surface in the proof of Lemma 5.5 in [17]. However, in our setting of multiplicity of convergence greater than two, one fixes an arbitrary compact domain $\Omega \subset \Sigma_\infty$ containing the origin and, for n sufficiently large, finds two domains $\Omega_1(n), \Omega_2(n) \subset \Sigma_n$, that are expressed as small normal oriented graphs over Ω , and have the same signed small constant mean curvatures. The proof of Lemma 5.5 then produces a positive Jacobi function F_Ω on Ω with $F_\Omega(\vec{0}) = 1$; since Ω is arbitrary, it follows that Σ_∞ would be stable, which we already showed is impossible. An analogous argument explains why, in the case the multiplicity of convergence is two, the similar geometric properties in item 4 above hold. Suppose that, for some compact unstable $\Omega \subset \Sigma_\infty$ containing the origin and, for n sufficiently large, one finds two domains $\Omega_1(n), \Omega_2(n) \subset \Sigma_n$ on the boundary of the collapsing region, that are expressed as small normal oriented graphs over Ω . If the mean curvature vector of the graph $\Omega_1(n)$ points toward $\Omega_2(n)$ (or vice versa) while the other points away from $\Omega_1(n)$, the argument is exactly the same as the previous one. If the mean curvature vectors of the graphs both point toward each other, then we can find a stable minimal surface in between them converging to Ω , giving again a contradiction because Ω is not stable. For still further details on this type of multiplicity of convergence at most two argument and for the collapsing description in item 4, see for example the proof of Case A of Proposition 3.1 in [22]. There it is explained, in a similar but slightly more general situation, that the convergence of a certain sequence of H_n -disks to a non-flat limit surface, which is a helicoid, has multiplicity at most two.

The multiplicity of convergence being at most two, together with a standard curve lifting argument, implies Σ_∞ has genus zero and that item 5 holds. Then the loop lifting property in item 5 implies that the flux of the limit properly embedded minimal surface Σ_∞ is zero. Since a minimal surface properly embedded in \mathbb{R}^3 with more than one end has nonzero flux [5], then Σ_∞ has one end. However, a complete genus zero surface with one end is simply-connected and so, Σ_∞ is a

helicoid by [17].

In summary, arbitrarily close to the origin, depending on the choice of $\bar{\varepsilon}$, there exist helicoid-like surfaces (rescalings of the surfaces Σ_n above) forming on M_n . Without loss of generality, after a translation of the M_n , we may assume that $p_n = \vec{0}$ and, abusing the notation, we will still assume that $\partial M_n \subset \partial \mathbb{B}(\varepsilon)$. In actuality ∂M_n lies on the boundary of a translation of $\partial \mathbb{B}(\varepsilon)$. The arguments in the following constructions would either remain the same or can be easily modified, if one desires to keep track of these translations.

Remark 3.1. After a possible rotation, we will also assume that Σ_∞ is a vertical helicoid containing the vertical x_3 -axis and the x_2 -axis.

The proof of Theorem 2.2 breaks up into the following two cases:

Case \mathcal{A} : The convergence of Σ_n to Σ_∞ has multiplicity one.

Case \mathcal{B} : The convergence of Σ_n to Σ_∞ has multiplicity two.

We will consider both **Case \mathcal{A}** and **Case \mathcal{B}** simultaneously. However, our constructions in **Case \mathcal{B}** will be based on using only the forming helicoids on the surfaces M_n that actually pass through the origin. In a first reading of the following proof, we suggest that the reader assume **Case \mathcal{A}** holds, as it is simpler to follow the constructions and the figures that we present in this case.

The Description 3.2 below follows from the smooth convergence of the Σ_n to Σ_∞ and Remark 3.1, because the statements in it hold for the limit vertical helicoid Σ_∞ that contains the x_3 -axis and x_2 -axis.

Description 3.2. Given $\varepsilon_2 \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$, there exists $\bar{\omega} > 0$ such that for any $\omega_1 > \omega_2 > \bar{\omega}$ there exist an $n_0 \in \mathbb{N}$ and positive numbers r_n , with $r_n = \frac{\sqrt{2}}{|A_{M_n}|(p_n)}$, such that for any $n > n_0$ the following statements hold. For clarity of exposition we abuse notation and we let $M = M_n$ and $r = r_n$.

1. $M \cap C(\omega_1 r, \pi(N+1)r)$ consists of either one disk component, if **Case \mathcal{A}** holds, or two disk components, if **Case \mathcal{B}** holds. One of the two possible disks in $M \cap C(\omega_1 r, \pi(N+1)r)$ contains the origin and we denote it by $M(\omega_1 r)$. If **Case \mathcal{B}** holds, we denote the other component by $M^*(\omega_1 r)$.
2. $M(\omega_1 r) \cap C(\omega_2 r, \pi(N+1)r)$ is also a disk and we denote it by $M(\omega_2 r)$.
3. For any $t \in [-(N+1)\pi r, (N+1)\pi r]$, $M(\omega_1 r)$ intersects the plane $\{x_3 = t\}$ transversely in a single arc and when t is an integer multiple of πr , this arc is disjoint from the solid vertical cylinder $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$. In particular,

$M(\omega_1 r) \cap C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is a collection of $2N + 2$ disks, each of which is a graph over

$$\{x_3 = 0\} \cap C(\omega_1 r, 1) \cap C\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right).$$

A similar description is valid for $M^*(\omega_1 r)$, if **Case B** holds.

4. $M(\omega_1 r) \cap [C(\omega_1 r, \pi(N + 1)r) - \text{Int}(C(\omega_2 r, \pi(N + 1)r))]$, that is

$$M(\omega_1 r) - \text{Int}(M(\omega_2 r)),$$

contains two oppositely oriented N -valued graphs u_1 and u_2 over $A(\omega_1 r, \omega_2 r)$. The boundary of each N -valued graph consists of four smooth arcs, as described in Definition 2.4 and the paragraph following it. Moreover, these graphs u_1 and u_2 can be chosen so that if **Case A** holds, then the related regions between the sheets satisfy

$$(\overline{W}[u_1] \cup \overline{W}[u_2]) \cap M = \text{graph}(u_1) \cup \text{graph}(u_2),$$

where $\overline{W}[u]$ denotes the closure of $W[u]$ in \mathbb{R}^3 . In other words, no other part of M comes between the sheets of $\text{graph}(u_1)$ and $\text{graph}(u_2)$.

If **Case B** holds, $M^*(\omega_1 r) \cap [C(\omega_1 r, \pi(N + 1)r) - \text{Int}(C(\omega_2 r, \pi(N + 1)r))]$ contains another pair of oppositely oriented N -valued graphs u_1^* and u_2^* over $A(\omega_1 r, \omega_2 r)$ and

$$\begin{aligned} & (\overline{W}[u_1] \cup \overline{W}[u_2] \cup \overline{W}[u_1^*] \cup \overline{W}[u_2^*]) \cap M = \\ & \text{graph}(u_1) \cup \text{graph}(u_2) \cup \text{graph}(u_1^*) \cup \text{graph}(u_2^*). \end{aligned}$$

5. The separation between the sheets of the N -valued graphs u_1 and u_2 is bounded; more explicitly, for $\rho_1, \rho_2 \in [\omega_2 r, \omega_1 r]$, $|\theta_1 - \theta_2| \leq 4\pi$ and $i = 1, 2$,

$$|u_i(\rho_1, \theta_1) - u_i(\rho_2, \theta_2)| < 6\pi r.$$

The same estimate is true for u_i^* , if **Case B** holds.

6. $|\nabla u_i| < \varepsilon_2$, $i = 1, 2$. The same estimate is true for u_i^* , if **Case B** holds.

For the sake of completeness, in the discussion below we provide some of the details that lead to the above description.

Let Σ_∞ be the vertical helicoid containing the x_2 and x_3 -axes with $|A_{\Sigma_\infty}|(\vec{0}) = \sqrt{2}$ and let Σ_n be as defined in the previous discussion. Then Σ_n converges smoothly with multiplicity one or two on compact subsets of \mathbb{R}^3 to Σ_∞ . Hence for

any $\omega_1 > \omega_2 > 0$ and $N \in \mathbb{N}$, each of the intersection sets $\Sigma_n \cap C(\omega_1, \pi(N+1))$ and $\Sigma_n \cap C(\omega_2, \pi(N+1))$ consist of either one or two disk components that satisfy the description in item 3 and 4, if n is taken sufficiently large. For simplicity, we provide some further details when **Case A** holds. If $\bar{\omega}$ is sufficiently large, given $\omega_1 > \bar{\omega}$, then $\Sigma_\infty \cap [C(\omega_1, \pi(N+1)) - \text{Int}(C(\bar{\omega}, \pi(N+1)))]$ contains two oppositely oriented N -valued graphs v_1 and v_2 over $A(\omega_1, \bar{\omega})$ such that

$$|v_i(\rho_1, \theta_1) - v_i(\rho_2, \theta_2)| < 5\pi, \quad i = 1, 2,$$

$$\rho_1, \rho_2 \in [\bar{\omega}, \omega_1], \quad |\theta_1 - \theta_2| \leq 4\pi,$$

and $|\nabla v_i| \leq \frac{\varepsilon_2}{2}$, $i = 1, 2$ and nothing else is trapped between the sheets of $\text{graph}(u_1)$ and $\text{graph}(u_2)$ in the sense made precise by the previous description. Given $\omega_2 \in (\bar{\omega}, \omega_1)$, because of the smooth convergence, there exists $\bar{n} \in \mathbb{N}$ such that for any $n > \bar{n}$, then

$$\Sigma_n \cap [C(\omega_1, \pi(N+1)) - \text{Int}(C(\omega_2, \pi(N+1)))]$$

contains two oppositely oriented N -valued graphs \tilde{u}_1 and \tilde{u}_2 over $A(\omega_1, \omega_2)$ such that for any $k = 1, \dots, N-1$,

$$|\tilde{u}_i(\rho_1, \theta_1) - \tilde{u}_i(\rho_2, \theta_2)| < 6\pi,$$

$$\rho_1, \rho_2 \in [\omega_2, \omega_1], \quad \theta_1, \theta_2 \in [(-N+2k-2)\pi, (-N+2k+2)\pi]$$

and $|\nabla \tilde{u}_i| \leq \varepsilon_2$, $i = 1, 2$. By definition $\Sigma_n = \frac{1}{\sqrt{2}}|A_{M_n}(\vec{0})|\widehat{M}_n$ and thus, rescaling proves that items 4, 5 and 6 hold.

Recall that a piecewise smooth domain in \mathbb{R}^3 is *mean convex*, if the mean curvature vectors of the smooth portions of the domain point into the domain and the interior angles at the non-smooth portions of its boundary are less than or equal to π .

For the next definition recall that $M = M_n$ for some $n \in \mathbb{N}$ large.

Definition 3.3. Define \widehat{X}_M to be the union of the closures of the (piecewise smooth) components of $\overline{\mathbb{B}}(\varepsilon) - M$ which have mean convex boundary.

The next lemma gives some information on the topology and geometry of $\widehat{X}_M \cap C(\omega_1 r, \pi(N+1)r)$.

Lemma 3.4. *The set $\widehat{X}_M \cap C(\omega_1 r, \pi(N+1)r)$ has one connected component X_M if **Case A** holds, or it has two connected components if **Case B** holds, and in this second case, we denote the component that contains $M(\omega_1 r)$ by X_M .*

Proof. If **Case A** holds, then $M \cap C(\omega_1 r, \pi(N+1)r)$ consists of a single properly embedded disk and as $C(\omega_1 r, \pi(N+1)r)$ is simply-connected, the lemma follows by elementary separation properties.

If **Case B** holds, then $M \cap C(\omega_1 r, \pi(N+1)r)$ consists of two disjoint disks, meaning that $\Sigma_n \cap C(\omega_1, \pi(N+1))$ consists of two disjoint disks, say D_1 and D_2 , for n large. Thus $C(\omega_1, \pi(N+1)) - \Sigma_n$ consists of three connected components. Let Ω be the component of $C(\omega_1, \pi(N+1)) - \Sigma_n$ such that $\partial\Omega = D_1 \cup D_2 \cup A$ where A is the annulus in $\partial C(\omega_1 r, \pi(N+1)r)$ with boundary $\partial D_1 \cup \partial D_2$. In order to prove the lemma, it suffices to show that the normal vectors to D_1 and D_2 point towards the exterior of Ω . This property follows from the earlier description that when the surfaces Σ_n converge with multiplicity two to the helicoid Σ_∞ , then the region between them collapses and the mean curvature vectors of the boundary surfaces of this region point away from it; here the region Ω corresponds to a part of this collapsing region. This finishes the proof of the lemma. \square

Remark 3.5. Suppose **Case B** holds. Let X'_M be a connected component of $X_M \cap C(\omega_1 r, \pi(N+1)r)$. For later reference, we note that by elementary separation properties, if γ is an open arc in $C(\omega_1 r, \pi(N+1)r)$ with endpoints $\partial\gamma \in \partial X'_M$ and $\gamma \cap \partial \hat{X}_M = \emptyset$, then either $\gamma \subset X'_M$ or $\gamma \subset [\mathbb{B}(\varepsilon) - \hat{X}_M]$.

4 Finding a minimal multi-valued graph on a fixed scale.

In what follows we wish to use the highly sheeted multi-valued graph forming on M near the origin to construct a minimal 10-valued graph forming near the origin that extends horizontally on a scale proportional to ε , and that lies on the mean convex side of $M = M_n$. Recall that $\varepsilon < \frac{1}{2}$. The planar domain M satisfies Description 3.2 for certain constants $\varepsilon_2 \in (0, \frac{1}{2})$, $\omega_1 > \omega_2 > \bar{\omega} > 0$, $r > 0$ and $N \in \mathbb{N}$ (for n sufficiently large). These constants will be finally fixed toward the end of this section. Nonetheless, in order for some of the statements to be meaningful, we will always assume N to be greater than $m+4$ where m is the number of boundary components of M . Recall that $M(\omega_1 r)$ and $M(\omega_2 r)$ are disks in M and that each disk resembles a piece of a scaled helicoid and contains the origin, c.f., Description 3.2.

Consider the intersection of

$$[\text{graph}(u_1) \cup \text{graph}(u_2)] \cap C\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right);$$

recall that $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is the truncated solid vertical cylinder of radius $\frac{1}{2}$, centered at $(\frac{1}{2} + \omega_2 r, 0, 0)$ with $|x_3| \leq 1$. This intersection consists of a collection

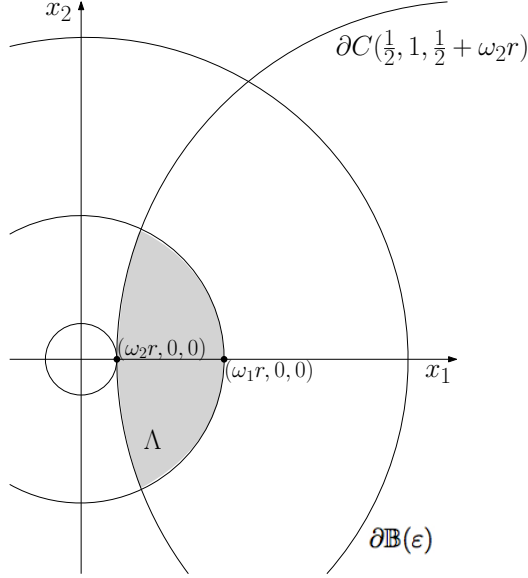


Figure 2:

of disk components

$$\Delta = \{\Delta_1, \dots, \Delta_{2N}\},$$

and each Δ_i is a graph over

$$\Lambda = \{x_3 = 0\} \cap C(\omega_1 r, 1) \cap C\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right),$$

see Figure 2.

Because M is embedded, the components of Δ can be assumed to be ordered by their relative vertical heights and then, by construction, the mean curvature vectors of consecutive components Δ_i and Δ_{i+1} have oppositely signed x_3 -coordinates. Without loss of generality, we will henceforth assume that the surface $\partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is in general position with respect to M ; this transversality hypothesis makes sense even though $\partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is non-smooth, because this surface is smooth at all points of intersection with M .

Let

$$\mathcal{F} = \{F(1), F(2), \dots, F(2N)\} \tag{1}$$

be the ordered listing of the components of $M \cap C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ that intersect the union of the disks in Δ , and that are indexed so that $\Delta_i \subset F(i)$ for each

$i \in \{1, 2, \dots, 2N\}$. Note that Δ_i and Δ_{i+j} , for some $j \in \mathbb{N}$, may be contained in the same component of $M \cap C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ and so, $F(i)$ and $F(i+j)$ may represent the same set. Observe that $\partial F(i) \subset \partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r) \cup \partial M$.

Property 4.1. 1. Suppose $i \in \{1, 2, \dots, 2N-1\}$. If $F(i) \cap \partial M = \emptyset$ and the mean curvature vector of $\Delta_i \subset F(i)$ is upward pointing, then $F(i+1) \cap \partial M \neq \emptyset$ or $F(i) = F(i+1)$.

2. Suppose $i \in \{2, 3, \dots, 2N\}$. If $F(i) \cap \partial M = \emptyset$ and the mean curvature vector of $\Delta_i \subset F(i)$ is downward pointing, then $F(i-1) \cap \partial M \neq \emptyset$ or $F(i) = F(i-1)$.

Proof. We will prove the first property; the proof of the second property is similar.

Suppose that $i \in \{1, \dots, 2N-1\}$ and the mean curvature vector of $\Delta_i \subset F(i)$ is upward pointing. Assume that $F(i+1) \cap \partial M = \emptyset = F(i) \cap \partial M$, and we will prove that $F(i) = F(i+1)$. Since $F(i) \cap \partial M = \emptyset$, then $\partial F(i) \subset \partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ and so $F(i)$ separates the simply-connected domain $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ into two connected domains. Since the top and bottom disks in $\partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ are disjoint from $\overline{\mathbb{B}}(\varepsilon)$ (because $\varepsilon < \frac{1}{2}$) and lie in the same component of $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r) - F(i)$ (because the top and bottom disks can be joined by a vertical segment in $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r) - \overline{\mathbb{B}}(\varepsilon)$), then the closure of one of these two connected domains, which we denote by $X(F(i)) \subset C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$, is disjoint from the top and bottom disks of the solid cylinder and it follows that $X(F(i)) \subset \overline{\mathbb{B}}(\varepsilon) \cap C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$.

For $t \geq 0$, consider the family of surfaces

$$\Omega_t = \partial C\left(\frac{1}{2}, 1, t + \frac{1}{2} + \omega_2 r\right).$$

The maximum principle for $H = 1$ surfaces applied to the family Ω_t shows that the last surface Ω_{t_0} which intersects $X(F(i))$, intersects $F(i) \subset \partial X(F(i))$ at an interior point of $F(i)$, where the mean curvature vector of $F(i)$ points into $X(F(i))$. Hence, $F(i)$ is mean convex when considered to be in the boundary of $X(F(i))$, see Figure 3.

Consider now a vertical line segment σ joining a point of the graph $\Delta_i \subset F(i)$ to a point of the graph $\Delta_{i+1} \subset F(i+1)$; note that by Lemma 3.4 and since the mean curvature vector of Δ_i is upward pointing, the interior of σ is disjoint from M , independently of whether **Case A** or **Case B** holds (see Remark 3.5). Since $X(F(i))$ is mean convex and the mean curvature vector of Δ_i is upward pointing, σ is contained in $X(F(i))$. Since $F(i+1) \cap \partial M = \emptyset$, then the similarly defined compact domain $X(F(i+1)) \subset C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ intersects $X(F(i))$ at the point

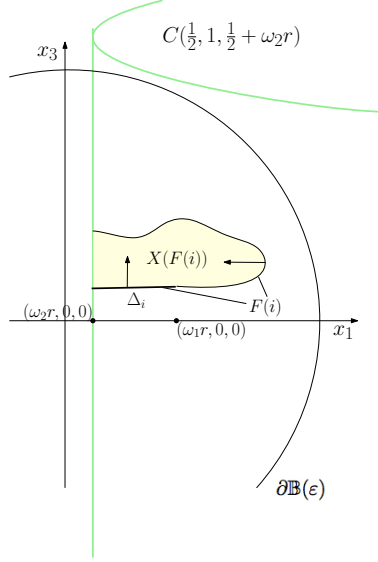


Figure 3:

$[\sigma \cap \Delta_{i+1}] \subset X(F(i+1))$ and so, since $F(i+1)$ is either equal to or disjoint from $F(i)$, then $X(F(i+1)) \subset X(F(i))$.

Since $X(F_i)$ intersects $X(F(i+1))$ at the point $[\sigma \cap \Delta_i] \subset X(F(i))$, the previous arguments imply $X(F(i)) \subset X(F(i+1))$. Because we have already shown $X(F(i+1)) \subset X(F(i))$, then $X(F(i)) = X(F(i+1))$, which implies $F(i) = F(i+1)$. This completes the proof. \square

Property 4.2. *There are at most $m - 1$ indices i , such that $F(i) = F(i+1)$ and $F(i) \cap \partial M = \emptyset$.*

Proof. Arguing by contradiction, suppose that there exist m increasing indices $\{i(1), i(2), \dots, i(m)\}$ such that for $j \in \{1, 2, \dots, m\}$,

$$F(i(j)) \cap \partial M = \emptyset \text{ and } F(i(j)) = F(i(j) + 1).$$

Note that for each $j \in \{1, 2, \dots, m\}$, $F(i(j)) \cap M(\omega_1 r)$ contains the disks $\Delta_{i(j)}$, $\Delta_{i(j)+1}$. Also note that since $F(i(j)) \cap \partial M = \emptyset$ for $j \in \{1, 2, \dots, m\}$, then $\cup_{j=1}^m \partial F(i(j)) \subset \partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$.

Let $\mathcal{F} = \{F(i(1)), F(i(2)), \dots, F(i(m))\}$ and let $F_1, \dots, F_{m'}$ be a listing of the distinct components in \mathcal{F} . For each $i = 1, \dots, m'$, let $n_i \geq 2$ denote the number of components of $F_i \cap M(\omega_1 r)$ and let d_i denote the number of times that

F_i appears in the list \mathcal{F} . Note that $n_i \geq d_i + 1$. Note that n_i might be strictly bigger than $d_i + 1$. Recall that by item 3 of Description 3.2, each component of $M(\omega_1 r) \cap C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ is a disk which intersects $\partial M(\omega_1 r)$ in a connected arc.

We next estimate the Euler characteristic of $M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))$ as follows.

Using that $\chi(F_i) \leq 1$ for each i and $\sum_{i=1}^{m'} d_i = m$ gives

$$\begin{aligned} \chi \left(M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j)) \right) &= \chi(M(\omega_1 r)) + \chi \left(\bigcup_{i=1}^{m'} F_i \right) - \chi \left(M(\omega_1 r) \cap \bigcup_{i=1}^{m'} F_i \right) \\ &= 1 + \sum_{i=1}^{m'} \chi(F_i) - \sum_{i=1}^{m'} n_i \\ &\leq 1 + m' - \sum_{i=1}^{m'} (d_i + 1) = 1 - m. \end{aligned}$$

Since the Euler characteristic of a connected, compact orientable surface is $2 - 2g - k$ where g is the genus and k is the number of boundary components, and since $M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))$ is a connected planar domain, which means $g = 0$, then the previous inequality implies that the number of boundary components of $M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))$ is at least $m + 1$. The hypothesis that for $j \in \{1, 2, \dots, m\}$, $F(i(j)) \cap \partial M = \emptyset$, implies that each boundary component of $M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))$ is disjoint from the boundary of M . Since M is a planar domain, each of these boundary components separates M , which implies that $M - [M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))]$ contains at least $m + 1$ components. Since M only has m boundary components, one of the components of $M - [M(\omega_1 r) \cup \bigcup_{j=1}^m F(i(j))]$, say T , is disjoint from ∂M . Note that $\partial T \subset C(\frac{1}{2}, 1, \frac{1}{2} - \omega_1 r)$ since $\partial T \subset [\partial M(\omega_1 r) \cup \bigcup_{j=1}^m \partial F(i(j))] \subset [\mathbb{B}(\varepsilon) \cap C(\frac{1}{2}, 1, \frac{1}{2} - \omega_1 r)]$.

Suppose for the moment that $\partial T \cap \partial M(\omega_1 r) \neq \emptyset$, and we will arrive at a contradiction. In this case, ∂T contains at least one point p in $\partial M(\omega_1 r) \cap \partial C(\frac{1}{2}, 1, \frac{1}{2} - \omega_1 r)$ and since T is disjoint from $M(\omega_1 r)$, T contains points outside $C(\frac{1}{2}, 1, \frac{1}{2} - \omega_1 r)$ near p . For $t \geq 0$, consider the family of translated surfaces

$$\Omega_t = \partial C \left(\frac{1}{2}, 1, \frac{1}{2} - \omega_1 r \right) + (-t, 0, 0).$$

Since the last such translated surface Ω_{t_0} which intersects T , does so at a point in the interior of T and T is contained on the mean convex side of Ω_{t_0} near this point, a standard application of the maximum principle for $H = 1$ surfaces gives a contradiction. This contradiction proves that $\partial T \cap \partial M(\omega_1 r) = \emptyset$.

Since we may now assume that $\partial T \cap \partial M(\omega_1 r) = \emptyset$, then

$$\partial T \subset \bigcup_{j=1}^m \partial F(i(j)) \subset \partial C\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right).$$

Let $p \in \partial T$. Since for some j , $p \in \partial F(i(j)) \subset \partial C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$ and also since T is disjoint from $F(i(j)) \subset C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$, then T contains points outside of $C(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r)$. A straightforward modification of the arguments in the previous paragraph using the maximum principle applied to the family of translated surfaces

$$\Omega_t = \partial C\left(\frac{1}{2}, 1, \frac{1}{2} + \omega_2 r\right) + (-t, 0, 0)$$

gives a contradiction. This contradiction completes the proof that Property 4.2 holds. \square

The next two propositions imply that if the N -valued graphs in M forming nearby the origin contain a sufficiently large number of sheets, then it is possible to find a disk in M whose boundary satisfies certain geometric properties. In the statement and proof of Property 4.3, we refer the reader to Figure 4.

Property 4.3. *Suppose that in the collection \mathcal{F} defined in (1), there exist $m + 1$ indexed domains*

$$F(i(1)), F(i(2)), \dots, F(i(m+1)),$$

not necessarily distinct as sets, with increasing indices and each of which intersects a fixed boundary component ν of ∂M and such that the mean curvature vectors of the associated pairwise disjoint subdomains $\Delta_{i(1)}, \Delta_{i(2)}, \dots, \Delta_{i(m+1)}$ are all upward pointing or all downward pointing. Then there exists a collection

$$\Gamma = \{\gamma(1), \gamma(2), \dots, \gamma(m+1)\}$$

of $m + 1$ pairwise disjoint, embedded arcs $\gamma(j) \subset F(i(j))$ with end points in $\partial F(i(j))$ such that:

1. *For each j , $\gamma(j) \cap M(\omega_1 r) = \gamma(j)$ is the arc $\Delta_{i(j)} \cap \{x_2 = 0\}$ and $\gamma(j)$ has the unique point p_j in $\Delta_{i(j)} \cap \partial C(\omega_2 r, 1)$ as one of its end points. The boundary of $\gamma(j)$ consists of p_j and a point in ν .*
2. *$M - (M(\omega_2 r) \cup \bigcup_{j=1}^{m+1} \gamma(j))$ contains a component whose closure is a disk D with ∂D consisting of an arc $\alpha \subset \nu$, two arcs in Γ and an arc β on the component τ of $\partial M(\omega_2 r) \cap \partial C(\omega_2 r, 1)$ that intersects $\Delta_{i(1)}$.*

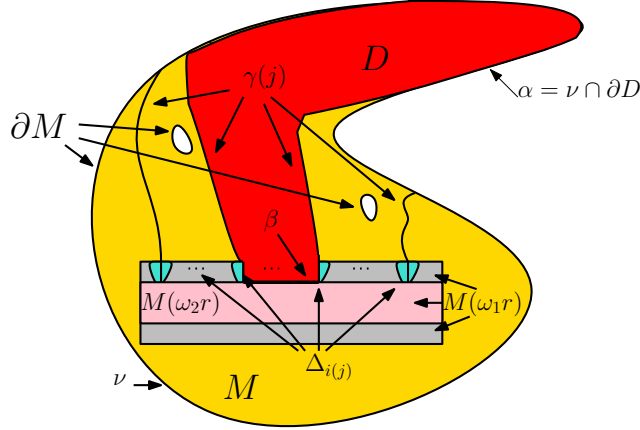


Figure 4: Here the number of boundary components of M is $m = 3$. The disks $\Delta_{i(1)}, \Delta_{i(2)}, \Delta_{i(3)}, \Delta_{i(4)}$ lie in the multigraph $u_1 \subset M(\omega_1 r) - \text{Int}(M(\omega_2 r))$ and are colored blue. The triple dots \cdots near β in the figure refer to a sequence of at least \tilde{N} of disks in Δ that intersect β but are not shown in the figure.

3. Furthermore, if for each $j \in \{1, 2, \dots, m\}$, $i(j+1) - i(j) \geq 2\tilde{N} + 2$ for some $\tilde{N} \in \mathbb{N}$, then $D \cap (M(\omega_1 r) - \text{Int}(M(\omega_2 r)))$ contains an \tilde{N} -valued graph over the annulus $A(\omega_1 r, \omega_2 r)$.

Proof. For each $j \in \{1, 2, \dots, m+1\}$, consider an embedded arc $\gamma(j)$ in $F(i(j))$ joining the point $p_j \in \Delta_{i(j)} \cap \partial C(\omega_2 r, \pi(N+1)r)$ to a point in ν ; this is possible since $F(i(j))$ intersects ν . Since $F(i(j)) - M(\omega_1 r)$ is path connected, then one can choose $\gamma(j)$ to intersect $C(\omega_1 r, \pi(N+1)r)$ in the arc $\Delta_{i(j)} \cap \{x_2 = 0\}$. As the arcs in $\Gamma = \{\gamma(1), \gamma(2), \dots, \gamma(m+1)\}$ can also be constructed to be pairwise disjoint, it is straightforward to check that item 1 holds for the collection Γ .

Since the mean curvature vectors of the domains $\Delta_{i(1)}, \dots, \Delta_{i(m+1)}$ are all upward or all downward pointing, without loss of generality we can assume that all of the points p_j lie on

$$\tau := \partial \text{graph}(u_1) \cap \partial C_{\omega_2 r},$$

that is the inner boundary of $\text{graph}(u_1)$.

Since M is a planar domain, $M - (M(\omega_2 r) \cup \Gamma)$ contains $m+1$ components. Because M has m boundary components, it follows that at least two of these $m+1$ components are disjoint from $\partial M - \nu$ and the closure of one of these components,

say D , intersects τ in a subarc β and it does not intersect $\partial M(\omega_2 r) - \beta$. It follows that D is a disk with boundary as described in item 2 of Property 4.3.

Item 3 follows immediately from the construction of D . \square

Proposition 4.4. *Suppose that $2N \geq m(2m+2)(m+2\tilde{N}+2)$. Then there exists a compact disk $D \subset M$ such that $D \cap (M(\omega_1 r) - \text{Int}(M(\omega_2 r)))$ contains an \tilde{N} -valued graph u^+ over the annulus $A(\omega_1 r, \omega_2 r)$. Moreover, there exist two indices i and j , with $i - j \geq 2\tilde{N} + 2$, such that ∂D consists of four consecutive arcs $\alpha, \sigma_1, \beta, \sigma_2$ satisfying:*

1. $\alpha = \partial D \cap \partial M \subset \partial \mathbb{B}(\varepsilon)$;
2. $\sigma_1 = \gamma(i)$ is an arc in $F(i)$ with end points in $\partial F(i)$, and such that $\gamma(i) \cap M(\omega_1 r) = \Delta_i \cap \{x_2 = 0\}$;
3. $\sigma_2 = \gamma(j)$ is an arc in $F(j)$ with end points in $\partial F(j)$, and such that $\gamma(j) \cap M(\omega_1 r) = \Delta_j \cap \{x_2 = 0\}$;
4. β is an arc in $\partial M(\omega_2 r) \cap \partial C(\omega_2 r, 1)$.

Proof. Since $\mathcal{F} = \{F(1), \dots, F(m(2m+2)(m+2\tilde{N}+2)), \dots, F(2N)\}$, then for $l \in \{1, 2, \dots, m(2m+2)\}$, the family of domains

$$T_l = \{F(i) \mid i \in \{(l-1)(m+2\tilde{N}+2)+1, \dots, (l-1)(m+2\tilde{N}+2)+m\}\}$$

is a well-defined subset of \mathcal{F} , each T_l consists of m , not necessarily distinct, indexed elements, and if $F(i) \in T_{l+1}$ and $F(j) \in T_l$, then $i - j \geq 2\tilde{N} + 3$. Properties 4.1 and 4.2 imply that there exists an element $F(f(l)) \in T_l$ such that $F(f(l)) \cap \partial M \neq \emptyset$. Thus, the collection

$$\mathcal{F}_1 = \{F(f(1)), F(f(2)), \dots, F(f(m(2m+2)))\}$$

has $m(2m+2)$ indexed elements and for each $l \in \{1, 2, \dots, m(2m+2)-1\}$, $f(l+1) - f(l) \geq 2\tilde{N} + 3$.

Since M has m boundary components, then there exists an ordered subcollection of \mathcal{F}_1

$$\mathcal{F}_2 = \{F(i(1)), F(i(2)), \dots, F(i(2m+2))\},$$

with each element in this ordered subcollection intersecting some particular component ν of ∂M . Therefore, there exists a further ordered subcollection of \mathcal{F}_2

$$\mathcal{F}_3 = \{F(k(1)), F(k(2)), \dots, F(k(m+1))\}$$

for which the mean curvature vectors of the disks $\Delta_{k(1)}, \Delta_{k(2)}, \dots, \Delta_{k(m+1)}$ are all upward pointing or all downward pointing. By construction, for each $j \in \{1, 2, \dots, m+1\}$, $k(j+1) - k(j) \geq 2\tilde{N} + 3$.

Proposition 4.4 now follows from Property 4.3 applied to the collection

$$\{F(k(1)), F(k(2)), \dots, F(k(m+1))\}.$$

□

Recall that X_M is the closure of the connected mean convex region of $\overline{\mathbb{B}}(\varepsilon) - M$ when **Case (A)** holds and it is the closure of the connected mean convex region of $\overline{\mathbb{B}}(\varepsilon) - M$ containing $M(\omega_1 r)$ when **Case (B)** holds. The next lemma is an immediate consequence of the main theorem in [25]. It says that because the domain X_M is mean convex, it is possible to find a stable embedded minimal disk in X_M with the same boundary as D , where D denotes the disk given in Proposition 4.4.

Lemma 4.5. *Let D denote the disk given in Proposition 4.4. Then there is a stable minimal disk E embedded in X_M with $\partial E = \partial D$.*

The next theorem, Theorem 4.6, shows that E contains a highly-sheeted multi-valued graph on a small scale near the origin and that some of the sheets of this multi-valued graph extend horizontally on a scale proportional to ε when \tilde{N} is sufficiently large, where \tilde{N} is described in Proposition 4.4.

Before stating and proving Theorem 4.6, we summarize some of the results that we have obtained so far. For simplicity, we suppose that **Case A** holds. Recall that M and r are elements of a sequence that depends on n and that for convenience we sometimes omit the index n .

This being the case, M_n separates $\overline{\mathbb{B}}(\varepsilon)$ into two components and X_M denotes the closure of the component of $\overline{\mathbb{B}}(\varepsilon) - M$ with mean convex boundary. Given $\varepsilon_2 \in (0, \frac{1}{2})$ and $\tilde{N} \in \mathbb{N}$, there exist $N \in \mathbb{N}$, $\bar{\omega} > 0$ such that for $\omega_1 > 5\omega_2 > \bar{\omega}$ there exist an $n_0 \in \mathbb{N}$ and positive numbers r_n , with $r_n = \frac{\sqrt{2}}{|A_{M_n}|(\bar{0})}$ converging to zero as $n \rightarrow \infty$, such that for any $n > n_0$ the following statements hold; see Description 3.2 for more details.

Again, for clarity of exposition we abuse the notations and we let $M = M_n$ and $r = r_n$.

1. $M \cap C(\omega_1 r, \pi(N+1)r)$ consists of the disk component $M(\omega_1 r)$ passing through the origin.
2. $M(\omega_1 r) \cap C(\omega_2 r, \pi(N+1)r)$ is also a disk, called $M(\omega_2 r)$.

3. $M(\omega_1 r) \cap [C(\omega_1 r, \pi(N+1)r) - \text{Int}(C(\omega_2 r, \pi(N+1)r))]$, that is

$$M(\omega_1 r) - \text{Int}(M(\omega_2 r)),$$

contains two oppositely oriented N -valued graphs u_1 and u_2 over $A(\omega_1 r, \omega_2 r)$ and

$$[\overline{W}[u_1] \cup \overline{W}[u_2]] \cap M = \text{graph}(u_1) \cup \text{graph}(u_2).$$

4. The separation between the sheets of the N -valued graphs u_1 and u_2 is bounded. In fact, for $\rho_1, \rho_2 \in [\omega_2 r, \omega_1 r]$, $|\theta_1 - \theta_2| \leq 4\pi$ and $i = 1, 2$,

$$|u_i(\rho_1, \theta_1) - u_i(\rho_2, \theta_2)| < 6\pi r.$$

5. $|\nabla u_i| < \varepsilon_2$, $i = 1, 2$.

Moreover, by Proposition 4.4 and Lemma 4.5, the N -valued graph u_1 contains an \tilde{N} -valued subgraph $\text{graph}(u^+)$, with u^+ defined over the same annulus $A(\omega_1 r, \omega_2 r)$ as u_1 and there exists an embedded stable minimal disk E disjoint from M whose boundary $\alpha \cup \sigma_1 \cup \beta \cup \sigma_2$ satisfies the following properties; see Figure 4:

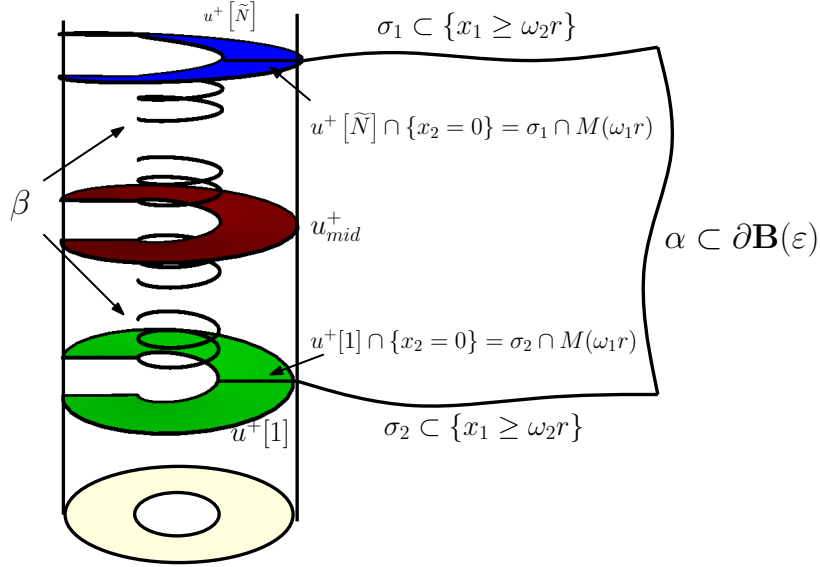


Figure 5:

- $\beta \subset \partial M(\omega_2 r) \cap \partial C_{\omega_2 r}$, the “inner” boundary, is an arc in the inner boundary of $\text{graph}(u^+)$, see Definition 2.4;
- $\alpha \subset \partial M \subset \partial \mathbb{B}(\varepsilon)$, the “outer” boundary;
- $\sigma_1 \cup \sigma_2 \subset \{x_1 \geq \omega_2 r\} \cap \{x_2 = 0\}$, the “side” boundaries;
- $M(\omega_1 r) \cap \sigma_1 = \{x_1 > 0, x_2 = 0\} \cap u^+[\tilde{N}]$ and $M(\omega_1 r) \cap \sigma_2 = \{x_1 > 0, x_2 = 0\} \cap u^+[1]$.

Recall that $u^+[k]$ denotes the k -th sheet of the \tilde{N} -valued graph u^+ ; see Definition 2.5. Recall also that N can be taken to be $m(m+1)(m+2\tilde{N}+2)$, where m is the number of boundary components of M ; see Proposition 4.4. Without loss of generality, we may assume that \tilde{N} is odd and the normal vector to u^+ is downward pointing and that u_2 contains an \tilde{N} -valued subgraph u^- satisfying the following properties:

1. $[\overline{W}[u^+] \cup \overline{W}[u^-]] \cap M = \text{graph}(u^+) \cup \text{graph}(u^-)$.
2. $u_{\text{mid}}^+ > u_{\text{mid}}^-$.

Furthermore after a small vertical translation of M by $(2\pi r)y$, for some $y \in [-2(N+2), \dots, 2(N+2)]$, we will assume u_{mid}^+ intersects $\{x_3 = 0\}$.

We are now ready to state Theorem 4.6.

Theorem 4.6. *Given $\tau \in (0, \frac{1}{2})$ there exists an $\Omega_1 = \Omega_1(\tau) > 1$, $\omega_2 = \omega_2(\tau)$ and $\omega_1 = \omega_1(\tau) > 10\omega_2$ such that the following is true of $M = M_n$ when n is sufficiently large.*

1. *There exists a minimal subdisk $E'_g \subset E$ which is a 10-valued graph E'_g over $A(\frac{\varepsilon}{\Omega_1}, 4\omega_2 r)$ with norm of the gradient less than τ .*
2. *Recall that $C_{\omega_1 r}$ is a closed infinite cylinder. The intersection $C_{\omega_1 r} \cap M \cap W[E'_g]$ consists of exactly two (when **Case A** holds) or four (when **Case B** holds) 9-valued graphs with norms of the gradients less than $\frac{\tau}{2}$. (See Definition 2.6 for the definition of $W[E'_g]$.)*
3. *There exist constants $\beta_1, \beta_2 \in (0, 1]$ such that if $\tau < \frac{\beta_2}{1000}$, then for any $p = (x_1, x_2, 0)$ such that $x_1^2 + x_2^2 = (\frac{\varepsilon}{10\Omega_1(\tau)})^2$ the following holds. The intersection set*

$$C_{\beta_1 \alpha}(p) \cap M \cap W[E'_g]$$

with $\alpha = \frac{80\varepsilon\tau}{\Omega_1(\tau)\beta_2}$ is non-empty and it consists of at least eight connected components. Note that $C_{\beta_1 \alpha}(p)$ does NOT contain the x_3 -axis.

Proof. Let $N_1 = N_1(\tau)$, $\varepsilon_1 = \varepsilon_1(\tau)$ and $\Omega_1 = \Omega_1(\tau)$ be as given by Theorem A.2. The idea of the proof is to find $N, \omega_1, \omega_2, \varepsilon_2, r$ in the previous description such that a subdisk $E' \subset E$ satisfies the hypotheses of Theorem A.2, and thus extends horizontally on a fixed scale proportional to ε .

For the next discussion, refer to Figure 6. Fix $\tilde{N} = N_1 + 5$ and consider the

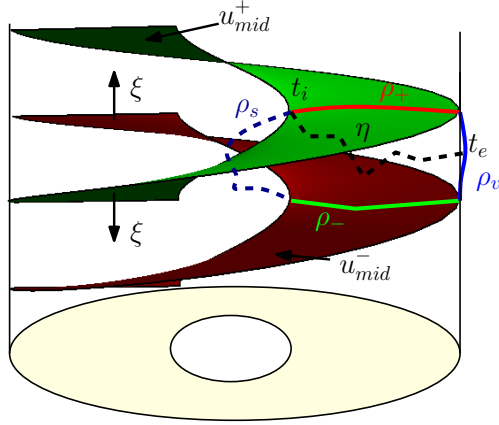


Figure 6:

simple closed curve $\rho = \rho_v \cup \rho_+ \cup \rho_s \cup \rho_-$ where

$$\rho_{\pm}: [\omega_2 r, 3\omega_2 r] \rightarrow \text{graph}(u_{\text{mid}}^{\pm}),$$

such that for $t \in [\omega_2 r, 3\omega_2 r]$,

$$\rho_{\pm}(t) = \{(t, 0, x_3) \mid x_3 \in (-\infty, \infty)\} \cap \text{graph}(u_{\text{mid}}^{\pm}).$$

The arc ρ_v is the open vertical line segment connecting $\rho_+(3\omega_2 r)$ and $\rho_-(3\omega_2 r)$. The arc ρ_s is an arc in $M(\omega_2 r)$ connecting $\rho_+(\omega_2 r)$ and $\rho_-(\omega_2 r)$; by item 3 in Description 3.2, ρ_s can be chosen to be contained in the slab $\{x_3(\rho_-(\omega_2 r)) \leq x_3 \leq x_3(\rho_+(\omega_2 r))\}$ and can be parameterized by its x_3 -coordinate. Note that by Description 3.2 and Lemma 3.4, $\rho_v \subset X_M$, $\rho_+ \cup \rho_s \cup \rho_- \subset \partial X_M$ and ρ is the boundary of a disk in $X_M \cap C_{3\omega_2 r}$. (This is also true if **Case B** holds.) Recall that $C_{3\omega_2 r}$ is the vertical solid cylinder centered at the origin of radius $3\omega_2 r$; see the definition at the beginning of this section. The main result in [25] implies that ρ is the boundary of an embedded least-area disk $D(\rho)$ in $X_M \cap C_{3\omega_2 r}$, which we may assume is transverse to the disk E . In particular, since by construction ρ intersects $\partial E = \alpha \cup \sigma_1 \cup \beta \cup \sigma_2$ transversely in the single point $t_i = u^+(\omega_2 r) \in \beta$, and

$\rho_+ \cup \rho_s \cup \rho_- \subset \partial X_M$, there exists an arc $\eta \subset D(\rho) \cap E$ with one end point t_i , the “interior” point, and its other end point $t_e \in \rho_v \cap E$, the “exterior” point.

Consider the connected component Γ_{t_e} of $C_{2\omega_2 r}(t_e) \cap E$ that contains t_e ; recall that $C_{2\omega_2 r}(t_e)$ is the solid vertical cylinder centered at t_e of radius $2\omega_2 r$. Then, Γ_{t_e} is contained in a slab of height less than $6\pi r$; see item 5 in Description 3.2. Applying Lemma A.1 with $h = \omega_2 r$ and $\beta h = 6\pi r$ gives that the connected component of $C_{\omega_2 r}(t_e) \cap \Gamma_{t_e}$ that contains t_e is a graph of a function with gradient bounded by $\frac{6\pi C_g}{\omega_2}$. After prolongating this graph, following the multi-valued graph $\text{graph}(u^+)$, we find that E contains an $(N_1 + 2)$ -valued graph E_g over $A(\omega_1 r - \omega_2 r, 3\omega_2 r)$ satisfying:

$$\text{the norm of the gradient of } E_g \text{ is less than } \frac{6\pi C_g}{\omega_2}.$$

The intersection set

$$\{(0, x_2, x_3) \mid x_2 > 0, x_3 \in (-\infty, \infty)\} \cap E_g$$

consists of $(N_1 + 2)$ arcs γ_i , $i = 1, \dots, N_1 + 2$, where the order of the indexes agrees with the relative heights of the arcs. Let γ_+ denote γ_{N_1+2} and let γ_- denote γ_1 . Let p_\pm^e denote the endpoint of γ_\pm in $\partial E_g \cap \partial C_{\omega_1 r - \omega_2 r}$ and let p_\pm^i denote the endpoint of γ_\pm in $\partial E_g \cap \partial C_{3\omega_2 r}$. Without loss of generality, we will assume that the plane $\{x_1 = 0\}$ intersects E transversally. Let Γ_+ denote the connected arc in $[E - \text{Int}(\gamma_+)] \cap \{x_1 = 0\}$ containing p_+^e and let Γ_- denote the connected arc in $[E - \text{Int}(\gamma_-)] \cap \{x_1 = 0\}$ containing p_-^e . Since $\Gamma_+ \cup \gamma_+$ and $\Gamma_- \cup \gamma_-$ are planar curves in the minimal disk E , neither of these curves can be closed in E by the maximum principle. This implies that $\partial\Gamma_+ = \{p_+^e, q_+\}$ and $q_+ \in \partial E$, $\partial\Gamma_- = \{p_-^e, q_-\}$ and $q_- \in \partial E$. Since the boundary of E is a subset of

$$\partial C_{\omega_2 r} \cup \{x_1 > \omega_2 r\} \cup \partial \mathbb{B}(\varepsilon)$$

and $\Gamma_\pm \subset \{x_1 = 0\}$, it follows that

$$q_\pm \in \partial E \cap [\partial C_{\omega_2 r} \cup \partial \mathbb{B}(\varepsilon)] = \alpha \cup \beta.$$

Claim 4.7. *The interiors of the arcs Γ_\pm are disjoint from $E_g \cup \eta$. Moreover, $q_\pm \in \alpha$.*

Proof. We will prove that $\Gamma_+ - \{p_+^e\}$ is disjoint from $E_g \cup \eta \cup \beta$, from which the claim follows for Γ_+ . The proof of the case for Γ_- is similar and will be left to the reader.

Arguing by contradiction, suppose $\Gamma_+ - \{p_+^e\}$ is not disjoint from $E_g \cup \eta \cup \beta$. Assuming that Γ_+ is parameterized beginning at p_+^e , let $r_+ \in \Gamma_+ \cap [E_g \cup \eta \cup \beta]$

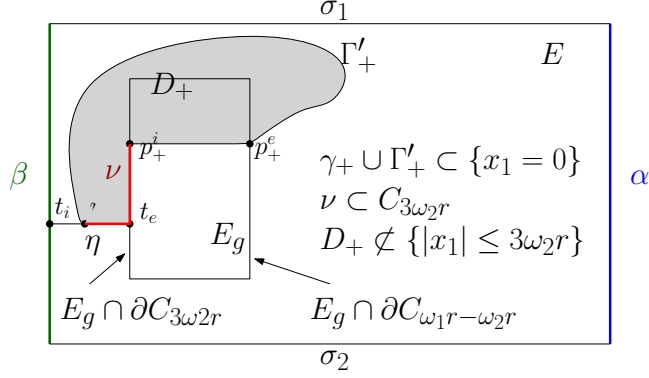


Figure 7: Here D_+ denotes the shaded region.

be the first point along Γ_+ in this intersection set and let Γ'_+ be the closed arc of $\Gamma_+ - \{r_+\}$ such that $\partial\Gamma'_+ = \{r_+, p_+^e\}$.

First suppose $r_+ \in \eta$, which is the case described in Figure 7. Then there exists an embedded arc

$$\nu \subset \eta \cup [E_g \cap \partial C_{3\omega_2 r}]$$

connecting r_+ and p_+^i . Recall that $E_g \cap \partial C_{3\omega_2 r}$ is the inner boundary of the multi-valued graph E_g . Therefore,

$$\Gamma = \Gamma'_+ \cup \nu \cup \gamma_+$$

is a simple closed curve that bounds a disk D_+ in E . By the convex hull property, D_+ is contained in the convex hull of its boundary. In particular, D_+ is contained in the vertical slab $\{|x_1| \leq 3\omega_2 r\}$. Note however that γ_+ separates the $(N_1 + 2)$ -th sheet of E_g , that is the set $u^+[N + 2]$, into two components that are not contained in the slab $\{|x_1| \leq 3\omega_2 r\}$ because $\omega_1 r > 4\omega_2 r$. By elementary separation properties, one of these components must be in $D_+ \subset \{|x_1| \leq 3\omega_2 r\}$, which gives a contradiction.

An argument similar to that presented in the previous paragraph shows $[\Gamma_+ - \{p_+^e\}] \cap E_g \cap \partial C_{3\omega_2 r} = \emptyset$ by using a similarly defined embedded arc $\nu \subset [E_g \cap \partial C_{3\omega_2 r}]$ connecting r_+ to p_+^i .

Next consider the case that $r_+ \in E_g \cap \partial C_{\omega_1 r - \omega_2 r}$. In this case, $r_+ \in \partial C_{\omega_1 r - \omega_2 r}$ is the end point of some component arc γ of $\{x_1 = 0\} \cap E_g$ with the other end point of γ being a point $p_\gamma \in \partial E_g \cap \partial C_{3\omega_2 r}$. Letting $\nu \subset \partial E_g \cap \partial C_{3\omega_2 r}$ be the arc with end points p_γ and p_+^i , we find that

$$\Gamma = \Gamma'_+ \cup \gamma \cup \nu \cup \gamma_+$$

is a simple closed curve that bounds a disk D_+ in E ; see Figure 8. Similar ar-

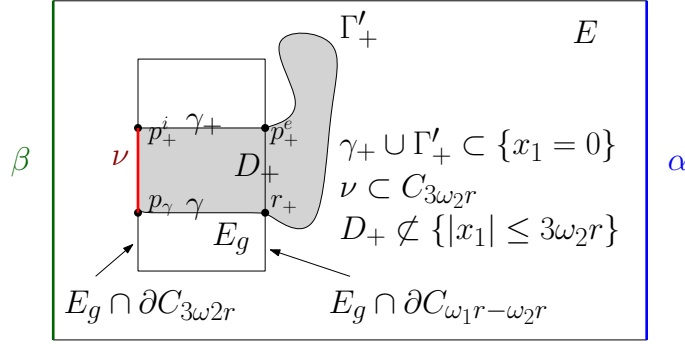


Figure 8:

guments as in the previous two paragraphs then provide the desired contradiction that the minimal disk D_+ cannot be contained in the convex hull of its boundary. Observe that $\gamma \cup \Gamma'_+ \cup \gamma_+ \subset \{x_1 = 0\}$ and $\nu \subset \partial C_{3\omega_2 r}$.

Finally, suppose that $r_+ \in \beta$. In this case, we find the desired simple closed curve

$$\Gamma = \Gamma'_+ \cup \nu \cup \gamma_+,$$

where

$$\nu \subset \beta \cup \eta \cup [E_g \cap \partial C_{3\omega_2 r}]$$

is an embedded arc that connects the points r_+ with p_+^i , see Figure 9. Arguing as previously, we obtain a contradiction and the claim is proved. \square

Denote by ξ_i the arc in $\partial E_g \cap \partial C_{3\omega_2 r}$ connecting p_+^i and p_-^i and let ξ_o be the arc in α connecting q_+ to q_- . The simple closed curve

$$\Gamma_+ \cup \gamma_+ \cup \xi_i \cup \gamma_- \cup \Gamma_- \cup \xi_o$$

bounds a subdisk E' in E which contains the N_1 -valued graph $E'_g = \bigcup_{k=2}^{N_1+1} u^+[k] \subset E_g$ over $A(\omega_1 r - \omega_2 r, 3\omega_2 r)$; see Figure 10. The norm of the gradient of this N_1 -valued graph E'_g is also less than $\frac{6\pi C_g}{\omega_2}$. Note that if $\omega_2 > \bar{\omega}_2 > 4\pi(N+2)$, where $\bar{\omega}_2$ is chosen sufficiently large, then

$$\partial E' \subset \mathbb{B}(4\omega_2 r) \cup \{x_1 = 0\} \cup \partial \mathbb{B}(\varepsilon).$$

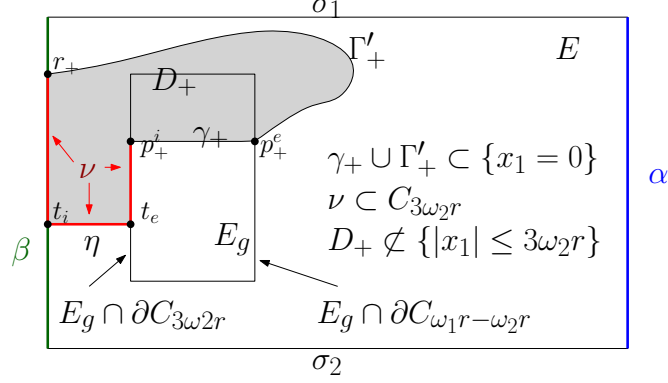


Figure 9:

The disk E' is the stable minimal disk to which we will apply Theorem A.2.

By construction E' contains the N_1 -valued subgraph E'_g of E'_g over the annulus $A(\omega_1 r - \omega_2 r, 4\omega_2 r)$ with norm of the gradient less than $\frac{6\pi C_g}{\omega_2}$. Thus, if $\omega_2 > \frac{6\pi C_g}{\varepsilon_1}$, E' satisfies item 2 of Theorem A.2 with

$$\delta = (\omega_1 - \omega_2)r, \quad \delta r_0 = 4\omega_2 r \quad \text{and} \quad r_0 = \frac{4\omega_2}{\omega_1 - \omega_2}.$$

Clearly, as described in Figure 10, there exists a curve $\tilde{\eta} \subset \Pi^{-1}(\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leq (\delta r_0)^2\})$ connecting \tilde{E}'_g to $\partial E' - \partial \mathbb{B}(\varepsilon)$ and thus E' also satisfies item 4 of Theorem A.2.

Recall that r depends on n and it goes to zero as $n \rightarrow \infty$. After choosing ω_1 sufficiently large, and then choosing n much larger, we obtain the inequalities

$$\frac{4\omega_2}{\omega_1 - \omega_2} \Omega_1 < 1 \quad \text{and} \quad \frac{\varepsilon}{r(\omega_1 - \omega_2)} \geq 1,$$

that together imply that the minimal disk E' satisfies item 1 of Theorem A.2.

Finally, let \tilde{u}_{mid} denote the middle sheet of \tilde{E}'_g . Using the bound $\varepsilon_1 := \frac{6\pi C_g}{\omega_2}$ for the norm of the gradient of \tilde{u}_{mid} and the fact that the inner spiral boundary of \tilde{u}_{mid} is contained in a small slab gives that

$$\Pi^{-1}(\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leq (\delta r_0)^2\}) \cap \tilde{u}_{\text{mid}} \subset \left\{ |x_3| \leq 20\pi\omega_2 r \cdot \frac{6\pi C_g}{\omega_2} \right\}$$

and if

$$\omega_2 > \frac{30\pi^2 C_g}{\varepsilon_1} \implies \frac{\pi C_g}{\omega_2} < \frac{\varepsilon_1}{30\pi}$$

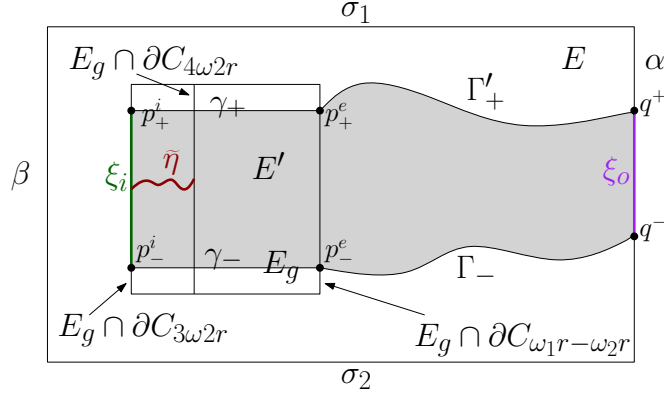


Figure 10: Here E' denotes the shaded region.

then $20\pi\omega_2 r \cdot \frac{6\pi C_g}{\omega_2} < \varepsilon_1 4\omega_2 r$; in other words, E' satisfies item 3 of Theorem A.2.

Recall that M and r depend on n and that $\bar{\omega}_2 > 4\pi(N+2)$ is chosen sufficiently large so that

$$\partial E' \subset \mathbb{B}(4\omega_2 r) \cup \{x_1 = 0\} \cup \partial \mathbb{B}(\varepsilon).$$

We now summarize what we have shown so far. If n is sufficiently large, then the quantities discussed at the beginning of the proof can be assumed to satisfy the following conditions:

$$\begin{aligned} \tilde{N} &= N_1 + 5, \quad \omega_2 > \max \left\{ \bar{\omega}_2, \frac{30\pi^2 C_g}{\varepsilon_1} \right\}, \\ \omega_1 &> 4\omega_2 \Omega_1 + 10\omega_2 \quad \text{and} \quad r < \frac{\varepsilon}{\omega_1 - \omega_2}, \end{aligned}$$

and then the stable minimal disk E' satisfies the hypotheses of Theorem A.2 and thus it contains a 10-valued graph E'_g over $A(\varepsilon/\Omega_1, 4\omega_2 r)$ with norm of the gradient less than τ . This completes the proof of item 1 in the theorem.

The proof that the intersection $C_{\omega_1 r} \cap M \cap W[E'_g]$ consists of exactly two or four 9-valued graphs follows from the construction of $W[E'_g]$ and Description 3.2; note that because of embeddedness and by construction, the multi-valued graphs on M near the origin and in $C_{\omega_1 r}$ spiral together with the minimal multivalued graph E'_g . Whether there are two or four 9-valued graphs depends on whether the convergence to the helicoid detailed in Description 3.2 is with multiplicity one or two. The norms of the gradients of such graphs are bounded by $\frac{\tau}{2}$ as long as ε_2 is, by item 6 of Description 3.2. This completes the proof of item 2 in the theorem.

The proof of item 3 will use the existence of the minimal 10-valued graph E'_g and a standard dragging argument. First note that using the gradient estimates for the minimal 10-valued graphs and the fact that the inner spiral boundary is contained in a slab of height $10 \cdot 2\pi\tau = 20\pi\tau < 80\tau$, then $W[E'_g]$ is contained in the open cone

$$\mathcal{C} = \left\{ (x_1, x_2, x_3) \mid |x_3| < 80\tau\sqrt{x_1^2 + x_2^2} \right\}.$$

We begin the proof of item 3 by proving the existence of certain embedded domains of vertical nodoids, where nodoids are the nonembedded surfaces of revolution of nonzero constant mean curvature defined by Delaunay [11] and where vertical means that the x_3 -axis is its axis of revolution.

Lemma 4.8. *There exist constants $\beta_1, \beta_2 \in (0, 1]$ such that the following holds. For $s \in (0, 1]$ consider the circles $C_1 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = \beta_1^2 s^2, x_3 = -\beta_2 s\}$ and $C_2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = \beta_1^2 s^2, x_3 = \beta_2 s\}$. Then there exists a subdomain \mathcal{N}_s of a vertical nodoid with constant mean curvature $\frac{1}{s}$ such that \mathcal{N}_s is embedded with boundary $C_1 \cup C_2$, and \mathcal{N}_s is contained in the convex hull of its boundary.*

Proof of Lemma 4.8. The lemma follows by defining $\mathcal{N}_s = s\mathcal{N}_1$, after finding the correct numbers β_1, β_2 for $s = 1$ and a compact embedded portion \mathcal{N}_1 of a vertical nodoid with constant mean curvature one and such that \mathcal{N}_1 is contained in the boundary of its convex hull. \square

Note that \mathcal{N}_s separates $C_{\beta_1 s}$ into a bounded and an unbounded component and its mean curvature vector points into the bounded component. Given $p \in \mathbb{R}^3$ we let $\mathcal{N}_s(p)$ be \mathcal{N}_s translated by p . Suppose now that $p = (x_1, x_2, 0)$ such that $x_1^2 + x_2^2 = (\frac{\varepsilon}{10\Omega_1(\tau)})^2$. Note that if $\tau < \frac{\beta_2}{1000}$, then $\mathcal{N}_\mu(p)$ with $\mu = \frac{80\varepsilon\tau}{\Omega_1(\tau)\beta_2}$ satisfies

$$\partial\mathcal{N}_\mu(p) \cap \mathcal{C} = \emptyset$$

and its mean curvature, $\frac{1}{\mu}$, is greater than one.

Consider the point $p = (\frac{\varepsilon}{10\Omega_1(\tau)}, 0, 0)$. Let

$$\Gamma = \left\{ (x_1, x_2, 0) : \left| \frac{x_2}{x_1} \right| < 2, x_1 > 0, (4\omega_2 r)^2 < x_1^2 + x_2^2 < \left(\frac{\varepsilon}{\Omega_1(\tau)} \right)^2 \right\}.$$

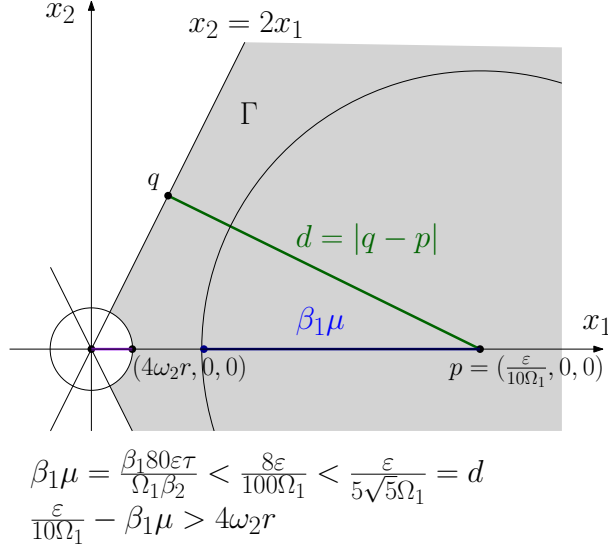


Figure 11: Γ is the grey region.

Note that E'_g separates $\Gamma \times \mathbb{R}$ into nine bounded components. Furthermore, without loss of generality, assuming that r is sufficiently small, we have that

$$\mathcal{N}_\mu(p) \subset \Gamma \times \mathbb{R},$$

see Figures 11 and 12.

Let $\Delta \subset \mathcal{C}$ be one of these bounded components; see Figure 12.

In order to prove item 3 of the theorem, we will first show that

$$\Delta \cap M \cap C_{\beta_1\mu}(p) \neq \emptyset.$$

To help understand the following arguments, see Figure 12. Since $\mathcal{N}_\mu(p) \subset C_{\beta_1\mu}(p)$, it suffices to show that $\Delta \cap M \cap \mathcal{N}_\mu(p) \neq \emptyset$. Consider the family of rescaled nodoids $\mathcal{N}^t = \mathcal{N}_{t\mu}(tp)$ for $t \in (0, 1]$. Since $\partial\mathcal{N}^t \cap \Delta = \emptyset$ and the mean curvature of \mathcal{N}^t is greater than one, by using a so-called dragging argument, it suffices to show that there exists some \bar{t} small so that $\Delta \cap M \cap \mathcal{N}^{\bar{t}}$ contains an interior point of $M \cap \Delta$ and $\mathcal{N}^{\bar{t}} \subset \Gamma \times \mathbb{R}$. This is because by an application of the mean curvature comparison principle, the family of nodoids \mathcal{N}^t , $t \in [\bar{t}, 1]$, cannot have a last point of interior contact with $M \cap \Delta$. Recall that $C_{\omega_1 r} \cap \Delta \cap M$ contains at least one component which is a graph over $\left\{ (x_1, x_2, 0) \mid \left| \frac{x_2}{x_1} \right| < 2, x_1 > 0, (4\omega_2 r)^2 < x_1^2 + x_2^2 < (\omega_1 r)^2 \right\}$ and

Theorem 5.1. *Given $\tau_2 > 0$, there exists $\Omega_2 = \Omega_2(\tau_2) \geq \Omega_1(\tau_2)$, where $\Omega_1(\tau_2)$ is given by Theorem A.2, and $\omega_2 = \omega_2(\tau_2)$ such that for n sufficiently large, the surface M_n contains two oriented 3-valued graphs G_n^{up}, G_n^{down} over $A(\varepsilon/\Omega_2, 4\omega_2 \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})})$ with norm of the gradient less than τ_2 , where G_n^{up} is oriented by an upward pointing normal and G_n^{down} is oriented by a downward pointing normal. Furthermore, these 3-valued graphs can be chosen to lie between the sheets of the 10-valued minimal graph $E'_g(n)$ given in Theorem 4.6 and so that $G_n^{up} \cap W[G_n^{down}]$ is a 2-valued graph.*

Proof. Recall that after normalizing the surfaces M_n by rigid motions that are expressed as translations by vectors of length at most $\bar{\varepsilon}$ for any particular small choice $\bar{\varepsilon} \in (0, \varepsilon/4)$, composed with rotations fixing the origin, we may assume (without loss of generality) that the surfaces M_n satisfy $|A_{M_n}|(\vec{0}) > n^2$ and that the origin is a point of almost-maximal curvature on M_n around which one or two vertical helicoids are forming in M_n on the scale of $|A_{M_n}|(\vec{0})$. It is in this situation that we apply Theorem 4.6 to obtain the 10-valued minimal graph $E'_g(n)$ described in the statement of Theorem 5.1.

By Theorem 4.6, for each $l \in \mathbb{N}$, there exist

$$\bar{n}(l) > 2, \quad \Omega_1(l) > 1, \quad \omega_2(l) > 0, \quad \omega_1(l) > 10\omega_2(l)$$

such that for $n > \bar{n}(l)$, E_n and thus M_n contains a minimal 10-valued graph $E'_g(n, l)$ over

$$A\left(\frac{\varepsilon}{\Omega_1(l)}, 4\omega_2(l) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})}\right)$$

with the norms of the gradients bounded by $\frac{1}{l}$; we will also assume for all $l \in \mathbb{N}$ that $\bar{n}(l+1) > \bar{n}(l) \in \mathbb{N}$ and that, after replacing by a subsequence and reindexing, the inequality

$$4\omega_2(l) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})} < \frac{\varepsilon}{n\Omega_1(l)}$$

also holds when $n > \bar{n}(l)$; in particular, under this assumption the ratios of the outer radius to the inner radius of the annulus over which the 10-valued minimal multigraph $E'_g(n, l)$ is defined go to infinity as n goes to infinity, and

$$\lim_{n \rightarrow \infty} 4\omega_2(l) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})} = 0.$$

Furthermore, by item 2 of Theorem 4.6, for $n > \bar{n}(l)$, the intersection

$$C_{\omega_1(l) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})}} \cap M_n \cap W[E'_g(n, l)]$$

consists of exactly two (**Case A**) or four (**Case B**) 9-valued graphs with the norms of the gradients bounded by $\frac{1}{2l}$.

Let

$$W_{n,l} := \{p \in \overline{W}[E'_g(n,l)] \mid -3\pi \leq \theta \leq 3\pi\},$$

where we are using cylindrical coordinates to parameterize $\overline{W}[E'_g(n,l)]$; see the last sentence in Definition 2.6. Let

$$\Pi_{n,l}: W_{n,l} \rightarrow \mathbb{R}_{>0} \times [-3\pi, 3\pi]$$

denote the “natural” projection. Given $p \in W_{n,l}$, the first component of $\Pi_{n,l}(p)$ is the horizontal radius of p , and its second component is the polar angle of the closest point to p in $E'_g(n,l)$ that lies below p and on the same vertical.

Then, for $n > \bar{n}(l)$, $C_{\omega_1(l) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})}} \cap M_n \cap W_{n,l}$ consists of a collection $\mathcal{C}_{n,l}$ of either two or four 3-valued graphs with the norms of the gradients bounded from above by $\frac{1}{2l}$.

Given $\tau > 0$, we claim that for some $l_\tau \in \mathbb{N}$ sufficiently large, and given $n > \bar{n}(l_\tau)$, then the 3-valued graphs in \mathcal{C}_{n,l_τ} extend horizontally to 3-valued graphs over

$$A\left(\frac{\varepsilon}{\bar{n}(l_\tau)\Omega_1(l_\tau)}, 4\omega_2(l_\tau) \frac{\sqrt{2}}{|A_{M_n}|(\vec{0})}\right)$$

with the norms of the gradients less than τ . This being the case, define G_n^{up}, G_n^{down} to be the two related extended graphs which have their normal vectors pointing up or down, respectively. Then, with respect these choices, the remaining statements of the theorem can be easily verified to hold.

Hence, arguing by contradiction suppose that the claim fails for some $\tau > 0$. Then for every $l \in \mathbb{N}$ sufficiently large, there exists a surface $M_{n(l)}$ with $n(l) > \bar{n}(l)$ such that the following statement holds: For

$$r_{n(l)} = \frac{\sqrt{2}}{|A_{M_{n(l)}}|(\vec{0})},$$

the 3-valued graphs in $\mathcal{C}_{n(l),l}$ do not extend horizontally as 3-valued graphs over the annulus

$$A\left(\frac{\varepsilon}{n(l)\Omega_1(l)}, 4\omega_2(l)r_{n(l)}\right)$$

with the norms of the gradients less than τ , where by our previous choices,

$$4\omega_2(l)r_{n(l)} < \frac{\varepsilon}{n(l)\Omega_1(l)} < \frac{\varepsilon}{2\Omega_1(l)}.$$

Thus for any fixed l large enough so that $M_{n(l)}$ exists, let

$$\rho(n(l)) \in [4\omega_2(l)r_{n(l)}, \frac{\varepsilon}{2\Omega_1(l)}]$$

be the supremum of the set of numbers $\rho \in [4\omega_2(l)r_{n(l)}, \frac{\varepsilon}{2\Omega_1(l)}]$ such that for any point $p \in C_\rho \cap M_{n(l)} \cap W_{n(l),l}$, the tangent plane to $M_{n(l)}$ at p makes an angle less than $\tan^{-1}(\tau)$ with the (x_1, x_2) -plane. Note that $\rho(n(l)) \geq \omega_1(l)r_{n(l)}$ because of the aforementioned properties of the surfaces in $\mathcal{C}_{n(l),l}$.

Let

$$\Pi_{n(l)} : W_{n(l),l} \rightarrow [4\omega_2(l)r_{n(l)}, \varepsilon/\Omega_1(l)] \times [-3\pi, 3\pi]$$

denote the natural projection as previously defined. The map $\Pi_{n(l)}$ restricted to $\text{Int}(C_{\rho(n(l))}) \cap M_{n(l)} \cap W_{n(l),l}$ is a proper submersion and thus the preimage of a sufficiently small neighborhood of a point

$$(\rho, \theta) \in [4\omega_2(l)r_{n(l)}, \rho(n(l))] \times [-3\pi, 3\pi]$$

consists of exactly two or four graphs. Let

$$p_{n(l)} = (\rho(n(l)), \theta_{n(l)}, x_3^{n(l)}) \in W_{n(l),l} \cap M_{n(l)}$$

be a point where the tangent plane of $M_{n(l)}$ makes an angle equal to $\tan^{-1}(\tau)$ with the (x_1, x_2) -plane and let $T_{n(l)}$ be the connected component of

$$M_{n(l)} \cap \mathbb{B}\left(p_{n(l)}, \frac{\rho(n(l))}{2}\right)$$

containing $p_{n(l)}$. Because of the gradient estimates for the 10-valued minimal graph $E'_g(n(l), l)$, $T_{n(l)}$ is contained in a horizontal slab of height at most $20\rho(n(l))\frac{1}{l}$. Furthermore, we remark that $\partial T_{n(l)} \subset \partial \mathbb{B}(p_{n(l)}, \frac{\rho(n(l))}{2})$ and $\Pi_{n(l)}$ restricted to

$$\text{Int}(C_{\rho(n(l))}) \cap T_{n(l)} \cap W_{n(l),l},$$

is at most four-to-one.

For each $n(l)$, consider the rescaled sequence $\tilde{T}_{n(l)} = \frac{1}{\rho_{n(l)}}T_{n(l)}$. We claim that the number of boundary components of $T_{n(l)}$, and thus of $\tilde{T}_{n(l)}$, is bounded from above by the number of boundary components of $M_{n(l)}$ which is at most m . Otherwise, since $M_{n(l)}$ is a planar domain, there exists a component Λ of $M_{n(l)} - T_{n(l)}$ that is disjoint from $\partial M_{n(l)}$ and contains points outside the ball $\mathbb{B}(p_{n(l)}, \frac{\rho(n(l))}{2})$. Since the mean curvature $H \leq 1$ and $\varepsilon < \frac{1}{2}$, an application of the mean curvature comparison principle with spheres centered at $p_{n(l)}$ implies that Λ

contains points outside of $\overline{\mathbb{B}}(\varepsilon)$; otherwise, the surface Λ lies in a smallest closed ball $\overline{\mathbb{B}}(p_{n(l)}, s_0)$, where $s_0 \in (\frac{\rho(n(l))}{2}, \varepsilon]$ and so, each point of $\Lambda \cap \partial\mathbb{B}(p_{n(l)}, s_0) \neq \emptyset$ has mean curvature at least $\frac{1}{\varepsilon} > 1$, which is a contradiction. However, $\Lambda \subset M_{n(l)} \subset \overline{\mathbb{B}}(\varepsilon)$ and this contradiction proves the claim.

Note that $\partial\tilde{T}_{n(l)} \subset \partial\mathbb{B}(\frac{1}{\rho(n(l))}p_{n(l)}, \frac{1}{2})$, and that the constant mean curvatures of the surfaces $\tilde{T}_{n(l)}$ are going to zero as $l \rightarrow \infty$. We next apply some of the previous results contained in this paper, e.g. Theorem 4.6, to study the geometry of the planar domain $\tilde{T}_{n(l)}$ near the point $\frac{1}{\rho(n(l))}p_{n(l)}$, which by our choices lies in ∂C_1 .

The surfaces $\tilde{T}_{n(l)}$ are contained in horizontal slabs of height at most $\frac{20}{l}$. Moreover, there exist rigid motions $\mathcal{R}_{n(l)}$ that are each a translation composed with a rotation around the x_3 -axis, such that the following hold:

1. $\mathcal{R}_{n(l)}(\frac{1}{\rho(n(l))}p_{n(l)}) = \vec{0}$.
2. For $\Gamma = \left\{ (x_1, x_2, 0) \mid x_1 > 0, x_1^2 + x_2^2 < \frac{1}{4}, x_2 < \frac{x_1}{4} \right\}$,
 $(\Gamma \times \mathbb{R}) \cap \mathcal{R}_{n(l)}(\tilde{T}_{n(l)})$ consists of at least two and at most four components, each of which is graphical over Γ . Note that in addition to other properties, Γ and $\mathcal{R}_{n(l)}$ are chosen so that $\Pi_{n(l)}[\rho(n(l))\mathcal{R}_{n(l)}^{-1}(\Gamma)] \subset [C_{\rho(n(l))} \cap \{x_3 = 0\}]$.

Let $\overline{T}_{n(l)} = \mathcal{R}_{n(l)}(\tilde{T}_{n(l)})$ and note that $\partial\overline{T}_{n(l)} \subset \partial\mathbb{B}(\frac{1}{2})$.

Since the height of the slab containing $\overline{T}_{n(l)}$ is going to zero as $l \rightarrow \infty$ and the tangent plane at $\vec{0}$ makes an angle of at least $\tan^{-1}(\tau)$ with the (x_1, x_2) -plane, it follows that as l goes to infinity, the norm of the second fundamental form of $\overline{T}_{n(l)}$ is becoming arbitrarily large at certain points in $\overline{T}_{n(l)}$ converging to $\vec{0}$. Using this property that as l goes to infinity, the norm of the second fundamental form of $\overline{T}_{n(l)}$ is becoming arbitrarily large nearby $\vec{0}$, we will prove that $\overline{T}_{n(l)}$ must intersect the region $\Gamma \times \mathbb{R}$ in more than four components, which will produce the desired contradiction.

After replacing by a subsequence and normalizing the surfaces by translating by vectors $\vec{v}_{n(l)}, \vec{v}_{n(l)} \rightarrow \vec{0}$, Theorem 4.6 implies that there exist a fixed rotation \mathcal{R} and constants $\beta_1, \beta_2 \in (0, 1]$ such that the following holds: Given $\tau_1 < \frac{\beta_2}{1000}$ there exists $\lambda_{\tau_1} \in \mathbb{N}$, $\omega_2(\tau_1)$ and $\Omega_1(\tau_1)$ such that for $l > \lambda_{\tau_1}$:

1. There exists a 10-valued minimal graph $E'_g(l, \tau_1)$ over

$$A\left(\frac{1}{2\Omega_1(\tau_1)}, 4\omega_2(\tau_1)\frac{\sqrt{2}}{|A_{\mathcal{R}(\overline{T}_{n(l)})}|(\vec{0})}, \right)$$

with norm of the gradient less than τ_1 (item 1 of Theorem 4.6);

2. With $\alpha = 40 \frac{1}{\Omega_1(\tau_1)} \frac{\tau_1}{\beta_2}$ and $p = (x_1, x_2, 0)$ with $x_1^2 + x_2^2 = (\frac{1}{20\Omega_1(\tau_1)})^2$, the intersection

$$C_{\beta_1\alpha}(p) \cap \mathcal{R}(\overline{T}_{n(l)}) \cap W[E'_g(l, \tau_1)].$$

is non-empty and contains at least eight connected components (item 3 of Theorem 4.6).

We claim that \mathcal{R} is a rotation around the x_3 -axis. Arguing by contradiction, suppose that this is not the case. Note that for any $\tau_1 < \frac{\beta_2}{1000}$, as l goes to infinity, the slab containing $\mathcal{R}(\overline{T}_{n(l)})$ converges to a plane \mathcal{P} through the origin that is not the (x_1, x_2) -plane as the slab was horizontal before applying \mathcal{R} . Therefore $E'_g(l, \tau_1)$ converges to a disk in \mathcal{P} . Let θ denote the angle that \mathcal{P} makes with the (x_1, x_2) -plane and pick

$$\tau_1 = \min(\tan(\theta/2), \frac{\beta_2}{1000}).$$

This choice of τ_1 leads to a contradiction because $E'_g(l, \tau_1)$ cannot converge as a set to a disk in \mathcal{P} and have norms of the gradients bounded by τ_1 . This contradiction proves that the rotation \mathcal{R} is a rotation around the x_3 -axis.

Finally, one obtains a contradiction by finding $p = (x_1, x_2, 0)$ with $x_1^2 + x_2^2 = (\frac{1}{20\Omega_1(\tau_1)})^2$ and taking τ_1 sufficiently small such that the disk centered at p of radius $\beta_1\alpha$ is contained in $\mathcal{R}(\Gamma)$. This leads to a contradiction because on the one hand, $C_{\beta_1\alpha}(p) \cap \mathcal{R}(\overline{T}_{n(l)})$ consists of at least eight components. On the other hand $\mathcal{R}(\Gamma \times \mathbb{R}) \cap \mathcal{R}(\overline{T}_{n(l)})$ consists of at most four components, each of which is graphical. This last contradiction completes the proof of Theorem 5.1. \square

The proof of item 1-4 of Theorem 2.7 follows from the geometric description we have provided for the elements of the sequence of surfaces M_n in this and previous sections. The proof of item 5 of Theorem 2.7 holds as well because as $|A_{M_n}|(\vec{0})$ becomes arbitrarily large, the number of sheets of the (vertical) helicoid forming nearby the origin also becomes arbitrarily large and one can apply the arguments described so far to any fixed number of different (horizontal) sections of the helicoid.

6 The Extrinsic Curvature and Radius Estimates for H -disks.

In this section we prove extrinsic curvature and radius estimates for H -disks that depend on the nonzero value of H ; in the Section 1.1 we show there are curvature

estimates only depend on a lower bound H_0 for H . The extrinsic curvature estimate will be a consequence of the Extrinsic Curvature Estimate for Planar Domains given in Theorem 2.2, once we prove that an H -disk with $H \leq 1$ and with boundary contained outside $\overline{\mathbb{B}}(R)$, where $R \leq \frac{1}{2}$, cannot intersect $\mathbb{B}(R)$ in a component with an arbitrarily large number of boundary components; see the next proposition for the existence of this bound. Also see Remark 2.3, where Σ can be thought of as the scaled surface $2\varepsilon M$ with M as in the next proposition.

Proposition 6.1. *There exists $N_0 \in \mathbb{N}$ such that for any $R \leq \frac{1}{2}$ and $H \leq 1$, if M is a compact disk of constant mean curvature H (possibly $H = 0$) with $\partial M \subset \mathbb{R}^3 - \mathbb{B}(R)$ and M is transverse to $\partial \mathbb{B}(R)$, then the closure of each component of $M \cap \mathbb{B}(R)$ is a smooth compact planar domain with at most N_0 boundary components.*

Furthermore, there exists an $\overline{R} \in (0, \frac{1}{2})$ such that whenever $R \leq \overline{R}$, then each component of $M \cap \overline{\mathbb{B}}(R)$ has at most 5 boundary components.

Proof. By the transversality hypothesis, we may assume without loss of generality that $\partial M \subset \mathbb{R}^3 - \overline{\mathbb{B}}(R)$. Since the surface M is transverse to the sphere $\partial \mathbb{B}(R)$, there exists a $\delta \in (0, \frac{R}{4})$ such that M intersects the closed δ -neighborhood of $\partial \mathbb{B}(R)$ in components that are smooth compact annuli, where each such component has one boundary curve in $\partial \mathbb{B}(R + \delta)$ and one boundary curve in $\partial \mathbb{B}(R - \delta)$, and such that each of the spheres $\partial \mathbb{B}(R + t)$ intersects M transversely for $t \in [-\delta, \delta]$. For the remainder of this proof we fix this value of δ , which depends on M .

If M is minimal, then the convex hull property implies N_0 can be taken to be 1. Assume now that M has constant mean curvature H , $H \in (0, 1]$.

Let Σ be a component of $M \cap \overline{\mathbb{B}}(R)$ with boundary curves

$$\Delta = \{\beta, \beta_1, \beta_2, \dots, \beta_n\}.$$

Here β denotes the boundary curve of Σ which is the boundary of the annular component of $M - \Sigma$, or equivalently, β is one of the two boundary curves of the component of $M - \Sigma$ that has ∂M in its boundary; see Figure 13 Left.

Let E be one of the two closed disks in $\partial \mathbb{B}(R)$ with boundary β . Let D_β be the open disk in M with boundary β and note that $\Sigma \subset D_\beta$. Next consider the piecewise-smooth immersed sphere $D_\beta \cup E$ in \mathbb{R}^3 and suppose that $D_\beta \cap \text{Int}(E)$ is a collection of k simple closed curves, which is not necessarily a subset of Δ . Then, after applying k surgeries to this sphere in the open δ -neighborhood of E , we obtain a collection of $(k + 1)$ pairwise disjoint piecewise-smooth embedded spheres; for the after-surgery picture when $k = 1$, see Figure 13 Right. We can assume that each of these pairwise disjoint piecewise-smooth embedded spheres (two in Figure 13) contains a smooth compact connected subdomain in D_β such

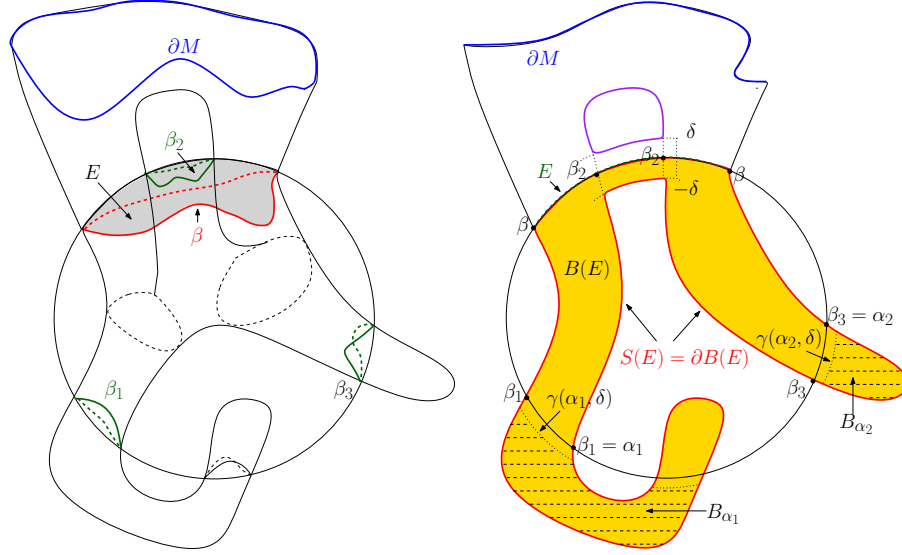


Figure 13: Left: Three dimensional view of M and E . Here the component Σ of $M \cap \mathbb{B}(R)$ has 4 boundary components $\beta, \beta_1, \beta_2, \beta_3$ and $\text{Int}(E) \cap M$ contains the simple closed curve β_2 . Right: A two-dimensional after-surgery schematic picture of M, Σ, E , the piecewise smooth yellow ball $B(E)$ and the simple closed curves $\gamma(\alpha_i, \delta) \subset S \cap \partial \mathbb{B}(R + \delta), i = 1, 2$. The shaded yellow regions denote $B_{\alpha_i} \subset [\mathbb{R}^3 - \mathbb{B}(R + \delta)], i = 1, 2$.

that the complement of this subdomain in the piecewise-smooth sphere consists of a finite number of disks contained in spheres of the form $\partial \mathbb{B}(R + t)$ for $t \in (-\delta, \delta)$.

Let $S(E)$ denote the sphere that contains E and lies in this collection of pairwise disjoint piecewise-smooth embedded spheres, let $\Omega(E)$ be the smooth compact connected subdomain $[D_\beta \cap S(E)] - \text{Int}(E)$ and let $B(E)$ denote the closed topological ball in \mathbb{R}^3 with boundary $S(E)$; see Figure 13.

Assertion 6.2. *Let Γ be the subcollection of curves in Δ which are not contained in E . The number of elements in Γ is bounded independently of $R \leq \frac{1}{2}$, $H \in (0, 1]$ and the choice of the component Σ . Furthermore, there exists an $\bar{R} \in (0, \frac{1}{2})$ such that if $R \leq \bar{R}$, then Γ has at most two elements.*

Proof. Assume that $\Gamma \neq \emptyset$ and we shall prove the existence of the desired bounds on the number of elements in Γ .

Recall the defining properties of $\delta \in (0, \frac{R}{4})$ given at the beginning of the proof of Proposition 6.1; in particular, each $\gamma \in \Gamma$ is a boundary component of a com-

compact annulus that is the intersection of $\Omega(E)$ with the compact region between the spheres $\partial\mathbb{B}(R)$ and $\partial\mathbb{B}(R + \delta)$. Hence, the condition $\Gamma \neq \emptyset$ implies there are points of $\Omega(E)$ of distance greater than $R + \delta$ from the origin, and so there is a point $p \in \text{Int}(\Omega(E))$ that is furthest from the origin. Since $S(E) - \Omega(E)$ is contained in $\mathbb{B}(R + \delta)$, p is also a point of $B(E)$ that is furthest from the origin. Since $B(E) \subset \overline{\mathbb{B}}(|p|)$, the mean curvature comparison principle implies that the mean curvature vector of $\Omega(E)$ points into $B(E)$ at p and that it also points towards the origin. Since $\Omega(E)$ is a smooth connected surface of positive mean curvature, it follows that all of the mean curvature vectors of $\Omega(E) \subset \partial B(E)$ point into $B(E)$.

Each $\alpha \in \Gamma$ bounds an open disk $D_\alpha \subset D_\beta$ which initially enters $\mathbb{R}^3 - \overline{\mathbb{B}}(R)$ near α and these disks form a pairwise disjoint collection. Each of these disks D_α intersects $\{x \in \overline{\mathbb{B}}(R + \delta) \mid |x| \geq R\} = \overline{\mathbb{B}}(R + \delta) - \mathbb{B}(R)$ in a subset that contains a smooth component that is a compact annulus with one boundary curve α and a second boundary component $\gamma(\alpha, \delta)$ in $\partial\mathbb{B}(R + \delta)$; see Figure 13 Right. Since $H \leq 1$ and $R \leq \frac{1}{2}$, the mean curvature comparison principle shows that the smooth component \widehat{D}_α of $S(E) \cap [\mathbb{R}^3 - \mathbb{B}(R + \delta)]$ containing $\gamma(\alpha, \delta)$, must contain a point p_α with $|p_\alpha| \geq \frac{1}{H} \geq 1$ of maximal distance from the origin. Also note that the mean curvature vector of \widehat{D}_α points towards the origin at p_α . Hence, since $B(E)$ is mean convex along $\Omega(E)$, points in $B(E)$ sufficiently close to p_α are contained in $\overline{\mathbb{B}}(|p_\alpha|)$. Once and for all, we make for each $\alpha \in \Gamma$, a particular choice for p_α if there is more than one possible choice.

Since $\mathbb{R}^3 - \overline{\mathbb{B}}(R + \delta)$ is simply-connected, elementary separation properties imply that for each $\alpha \in \Gamma$, $\mathbb{R}^3 - [\widehat{D}_\alpha \cup \overline{\mathbb{B}}(R + \delta)]$ contains two components, and let B_α be the closure of the bounded component; see Figure 13 Right.

By construction ∂B_α is an embedded piecewise-smooth compact surface, the domain $\widehat{D}_\alpha \subset \partial B_\alpha$ is connected and $\partial B_\alpha - \widehat{D}_\alpha$ consists of a finite number of precompact domains in $\partial\mathbb{B}(R + \delta)$. Since the point p_α is a point of B_α of maximal distance from the origin, the inward pointing normal of ∂B_α at p_α points toward the origin. Therefore \widehat{D}_α has positive mean curvature as part of the boundary of B_α oriented by its inward pointing normal.

We claim that $\{B_\alpha\}_{\alpha \in \Gamma}$ is a collection of indexed compact pairwise disjoint domains. To see this first notice that if $B_\alpha \cap B_{\alpha'} \neq \emptyset$ for some $\alpha, \alpha' \in \Gamma$ with $\alpha \neq \alpha'$, then one of the containments $B_\alpha \subset B_{\alpha'}$ or $B_{\alpha'} \subset B_\alpha$ holds, which implies that either $\widehat{D}_\alpha \subset B_{\alpha'}$ or $\widehat{D}_{\alpha'} \subset B_\alpha$, respectively. Therefore, to prove the claim it suffices to show that $S(E) \cap B_\alpha = \widehat{D}_\alpha$. Clearly, since for each $\alpha \in \Gamma$, $\widehat{D}_\alpha \subset S(E)$ and $\widehat{D}_\alpha \subset \partial B_\alpha$, then $\widehat{D}_\alpha \subset S(E) \cap B_\alpha$. If $S(E) \cap B_\alpha \not\subset \widehat{D}_\alpha$, then there exists a point $p \in [S(E) - \widehat{D}_\alpha] \cap B_\alpha$ that is furthest from the origin. By our earlier small positive choice of $\delta > 0$, the sphere $\partial\mathbb{B}(R + \delta)$ intersects M transversely and so the point p lies in the interior of B_α . Let r_p be the ray

$\{tp \mid t \geq 1\}$, let $t_0 > 1$ be the smallest $t > 1$ such that $tp \in \widehat{D}_\alpha$ and let γ be the open segment $\{tp \mid 1 < t < t_0\}$. Since $p \in [S(E) - \widehat{D}_\alpha] \cap B_\alpha$ is a point in this set that is furthest from the origin, the closed segment $\overline{\gamma}$ intersects $S(E)$ only at its end points, namely p and t_0p , which means that the segment γ is either in $B(E)$ or it is contained in the complement of $B(E)$.

On the one hand, since $D_\beta \cap S(E) \subset \partial B(E)$ has positive mean curvature as part of the boundary of $B(E)$ oriented by its inward pointing normal and the mean curvature vector of D_β at p points towards the origin, the ray γ enters the complement of $B(E)$ near p , which implies that $\gamma \subset [\mathbb{R}^3 - B(E)]$. On the other hand, \widehat{D}_α is mean convex as part of the boundary of $B(E)$ and also as part of the boundary of B_α . Since p lies in the interior of B_α , then $\gamma \subset B_\alpha$, which implies that γ is contained in the interior of $B(E)$ near $t_0p \in \widehat{D}_\alpha$. This is a contradiction which proves that $S(E) \cap B_\alpha = \widehat{D}_\alpha$. Hence, $\{B_\alpha\}_{\alpha \in \Gamma}$ is a collection of compact pairwise disjoint domains.

Although we did not subscript the collection $\{B_\alpha\}_{\alpha \in \Gamma}$ with the variable δ , the domains in it do depend on δ . Letting $\delta \rightarrow 0$, we obtain a related collection of limit compact domains $\{B_\alpha\}_{\alpha \in \Gamma}$, which we denote in the same way and which are pairwise disjoint. Let r_α be the ray $\{s \frac{p_\alpha}{|p_\alpha|} \mid s > 0\}$ and for $t \in (0, |p_\alpha|]$, let $\Pi(\alpha)_t$ be the plane perpendicular to r_α at the point $t \frac{p_\alpha}{|p_\alpha|}$. A standard application of the Alexandrov reflection principle to the region B_α and using the family of planes $\Pi(\alpha)_t$, gives that the connected component U_α of $\widehat{D}_\alpha - \Pi(\alpha)_{\frac{R+|p_\alpha|}{2}}$ containing p_α is graphical over its projection to $\Pi(\alpha)_{\frac{R+|p_\alpha|}{2}}$ and its image \widehat{U}_α , under reflection in $\Pi(\alpha)_{\frac{R+|p_\alpha|}{2}}$, is contained in B_α . Thus, if $\alpha_1, \alpha_2 \in \Gamma$ and $\alpha_1 \neq \alpha_2$, then $\widehat{U}_{\alpha_1} \cap \widehat{U}_{\alpha_2} = \emptyset$ because $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$.

Since $R \leq \frac{1}{2}$ and $|p_\alpha| \geq \frac{1}{H} \geq 1$, the point p_α has height at least $\frac{1-R}{2} \geq \frac{1}{4}$ over the plane $\Pi(\alpha)_{\frac{R+|p_\alpha|}{2}}$. Let $\widehat{p}_\alpha \in \widehat{U}_\alpha$ be the point in $\partial \mathbb{B}(R) \cap \partial B_\alpha$ that is the reflection of p_α in the plane $\Pi(\alpha)_{\frac{R+|p_\alpha|}{2}}$. By the uniform curvature estimates in [27] for oriented graphs with constant mean curvature (graphs are stable with curvature estimates away from their boundaries), it follows that each of the graphs \widehat{U}_α contains a disk $\widehat{F}(\alpha)$ that is a radial graph over a closed geodesic disk $D(\widehat{p}_\alpha, \varepsilon R)$ in $\partial \mathbb{B}(R)$ centered at \widehat{p}_α and of fixed geodesic radius $\varepsilon R > 0$, where ε is independent of M , R , α and $H \in (0, 1]$. Since the surfaces \widehat{U}_α form a pairwise disjoint collection of surfaces, the distances on $\partial \mathbb{B}(R)$ between the centers of different disks of the form $D(\widehat{p}_\alpha, \varepsilon R)$, $\alpha \in \Gamma$, must be greater than εR . Therefore, $\{D(\widehat{p}_\alpha, \frac{\varepsilon R}{2}) \mid \alpha \in \Gamma\}$ is a pairwise disjoint collection of disks in $\partial \mathbb{B}(R)$. Since a sphere of radius R contains a uniformly bounded number of pairwise disjoint geodesic disks of radius $\frac{\varepsilon R}{2}$, independent of R , the last observation implies the first statement in Assertion 6.2.

We now prove the second statement in the assertion. Arguing by contradiction, suppose M_n is a sequence of disks satisfying the conditions of Proposition 6.1, with mean curvatures $H_n \in [0, 1]$, $\partial M_n \subset \mathbb{R}^3 - \mathbb{B}(R_n)$ and $R_n \searrow 0$. Suppose that for each $n \in \mathbb{N}$, $M_n \cap \mathbb{B}(R_n)$ contains a component Σ_n so that Γ_n has at least three components $\alpha_1(n), \alpha_2(n), \alpha_3(n)$. Now replace the disks M_n by the scaled disks $\frac{1}{2R_n}M_n$ with mean curvatures $2R_n H_n$ converging to 0 as $n \rightarrow \infty$. For $k = 1, 2, 3$, a subsequence of the related sequence of stable constant mean curvature graphs $\widehat{U}_{\alpha_k}^n$ defined earlier converges to a flat plane Π_k tangent to $\partial \mathbb{B}(\frac{1}{2})$. Since the graphs $\widehat{U}_{\alpha_1}^n, \widehat{U}_{\alpha_2}^n, \widehat{U}_{\alpha_3}^n$ are pairwise disjoint, if R_n is sufficiently small, the number of these graphs must be at most two, otherwise after choosing a subsequence, two of the related limit planes Π_1, Π_2, Π_3 must be non-parallel and in this case one would find that the two related graphs $\widehat{U}_{\alpha_1}^n, \widehat{U}_{\alpha_2}^n, \widehat{U}_{\alpha_3}^n$ intersect for n sufficiently large. This contradiction completes the proof of Assertion 6.2. \square

Proposition 6.1 follows immediately from the estimates in Assertion 6.2. To see this, observe that there are two possible choices for the closed disk E appearing in the proof of Assertion 6.2 (both choices have the common boundary β). Therefore, there are also two possible choices Γ_1, Γ_2 for the subcollection $\Gamma \subset \Delta$ defined in the statement of Assertion 6.2 and $\Delta = \Gamma_1 \cup \Gamma_2 \cup \{\beta\}$. By Assertion 6.2 there exists a bound on the number of elements in each of the sets Γ_1, Γ_2 that is independent of $R \leq \frac{1}{2}$ and $H, H \in [0, 1]$; this proves the first statement in Proposition 6.1. Applying the ‘Furthermore’ part of Assertion 6.2 to each of the collections Γ_1, Γ_2 , we directly deduce the ‘Furthermore’ part of Proposition 6.1. \square

In the next lemma we prove curvature estimates for H -disks that depends on the nonzero value H of the mean curvature.

Lemma 6.3. *Given $\delta > 0$ and $H \in (0, \frac{1}{2\delta})$, there exists a constant $K_0(\delta, H)$ such that for any compact H -disk \mathcal{D} ,*

$$\sup_{\{p \in \mathcal{D} \mid d_{\mathbb{R}^3}(p, \partial \mathcal{D}) \geq \delta\}} |A_{\mathcal{D}}| \leq K_0(\delta, H).$$

Proof. Let \mathcal{D} be a compact H -disk as in the statement of the lemma, and suppose that there is a point $p \in \mathcal{D}$ such that $d_{\mathbb{R}^3}(p, \partial \mathcal{D}) \geq \delta$. After translating \mathcal{D} , we may assume that $p = \vec{0}$ and that $\partial \mathbb{B}(\delta)$ intersects \mathcal{D} transversally. By Proposition 6.1 applied to $M = \frac{1}{2\delta}\mathcal{D}$, there is a universal $N_0 \in \mathbb{N}$ such that the component Σ of $\mathcal{D} \cap \mathbb{B}(\delta)$ containing $\vec{0}$ has at most N_0 boundary components.

Since $\Sigma \subset \mathcal{D}$, the planar domain Σ has zero flux. After setting $\varepsilon = \delta$ and applying Theorem 2.2 to Σ (with $m = N_0$), we find that there is a constant $K_0(\delta, H)$ such that $|A_{\mathcal{D}}|(\vec{0}) \leq K_0(\delta, H)$, which proves the lemma. \square

7 The proof of Theorem 1.1.

The next theorem states that there exists an upper bound for the extrinsic distance from a point in an H -disk to its boundary.

Theorem 7.1 (Extrinsic Radius Estimates). *There exists a constant $\mathcal{R}_0 > 0$ such that any compact H -disk \mathcal{D} has extrinsic radius less than $\frac{\mathcal{R}_0}{H}$. In other words, for any point $p \in \mathcal{D}$,*

$$d_{\mathbb{R}^3}(p, \partial\mathcal{D}) < \mathcal{R}_0/H.$$

Proof. By scaling arguments, it suffices to prove the theorem for $H = 1$. Arguing by contradiction, suppose that the radius estimate fails. In this case, there exists a sequence of 1-disks \mathcal{D}_n passing through the origin such that for each n , $d_{\mathbb{R}^3}(\vec{0}, \partial\mathcal{D}_n) \geq n + 1$. Without loss of generality, we may assume that $\partial\mathbb{B}(n)$ intersects \mathcal{D}_n transversally. Let Δ_n be the smooth component of $\mathcal{D}_n \cap \mathbb{B}(n)$ with $\vec{0} \in \Delta_n$. By Lemma 6.3, the surfaces Δ_n have uniformly bounded norm of the second fundamental form. A standard compactness argument, see for instance Section 3 in this manuscript or the paper [23], gives that a subsequence of Δ_n converges with multiplicity one to a genus zero, strongly Alexandrov embedded¹ 1-surface Δ_∞ with bounded norm of the second fundamental form.

Since the genus of Δ_∞ is zero and it has bounded norm of the second fundamental form, then, by item 2 of Corollary 3.4 in [23], for some divergent sequence of points $q_n \in \Delta_\infty$, the translated surfaces $\Delta_\infty - q_n$ converge with multiplicity one to a strongly Alexandrov embedded surface $\tilde{\Delta}_\infty$ in \mathbb{R}^3 such that the component passing through $\vec{0}$ is an embedded Delaunay surface. Since a Delaunay surface has nonzero flux, we conclude that the original disks \mathcal{D}_n also have nonzero flux for n large, which is a contradiction. This contradiction proves that the extrinsic radius of a 1-disk \mathcal{D} is bounded by a universal constant, and Theorem 7.1 now follows. \square

Using Theorem 7.1, we now prove the extrinsic curvature estimates stated in the Introduction. For the reader's convenience, we restate Theorem 1.1 below.

Theorem 7.2 (Extrinsic Curvature Estimates). *Given $\delta, \mathcal{H} > 0$, there exists a constant $K_0(\delta, \mathcal{H})$ such that for any H -disk \mathcal{D} with $H \geq \mathcal{H}$,*

$$\sup_{\{p \in \mathcal{D} \mid d_{\mathbb{R}^3}(p, \partial\mathcal{D}) \geq \delta\}} |A_{\mathcal{D}}| \leq K_0(\delta, \mathcal{H}).$$

Proof of Theorem 7.2. Arguing by contradiction, suppose that the theorem fails for some $\delta, \mathcal{H} > 0$; without loss of generality we may assume that $\delta \leq \frac{1}{2}$ and $\mathcal{H} \leq 1$.

¹ Δ_∞ is the boundary of a properly immersed complete three-manifold $f: N^3 \rightarrow \mathbb{R}^3$ such that $f|_{\text{Int}(N^3)}$ is injective and $f(N^3)$ lies on the mean convex side of Δ_∞ .

In this case there exists a sequence of compact H_n -disks with $H_n \geq \mathcal{H}$ and points $p_n \in \mathcal{D}_n$ satisfying:

$$n \leq |A_{\mathcal{D}_n}|(p_n). \quad (2)$$

$$\delta \leq d_{\mathbb{R}^3}(p_n, \partial \mathcal{D}_n), \quad (3)$$

Rescale these disks by H_n to obtain the sequence of 1-disks $\widehat{\mathcal{D}}_n = H_n \mathcal{D}_n$ and a related sequence of points $\widehat{p}_n = H_n p_n$. By definition of these disks and points, and equations (3) and (2), we have that

$$\frac{n}{H_n} \leq |A_{\widehat{\mathcal{D}}_n}|(\widehat{p}_n), \quad (4)$$

$$\delta \mathcal{H} \leq \delta H_n \leq d_{\mathbb{R}^3}(\widehat{p}_n, \partial \widehat{\mathcal{D}}_n) \leq \mathcal{R}_0, \quad (5)$$

where \mathcal{R}_0 is the constant that appears in the statement of Theorem 7.1.

Equation (5), our two initial assumptions that $\delta \leq \frac{1}{2}$ and $\mathcal{H} \leq 1$ and Lemma 6.3 imply that

$$|A_{\widehat{\mathcal{D}}_n}|(\widehat{p}_n) \leq K_0(\delta \mathcal{H}, 1).$$

This inequality, together with equations (4) and (5), then gives

$$\frac{\delta}{\mathcal{R}_0} n \leq \frac{n}{H_n} \leq |A_{\widehat{\mathcal{D}}_n}|(\widehat{p}_n) \leq K_0(\delta \mathcal{H}, 1),$$

which gives a contradiction for n chosen sufficiently large. This contradiction proves the desired curvature estimate. \square

A Appendix.

The first result in this appendix is a gradient estimate for certain stable minimal surfaces in thin slabs, which follows from an application of the curvature estimates by Schoen [28] for stable orientable minimal surfaces.

Lemma A.1 (Lemma I.0.9. in [6]). *Let $\Gamma \subset \{|x_3| \leq \beta h\}$ be a compact stable embedded minimal surface and let $T_h(\Pi(\partial \Gamma)) \subset \mathbb{R}^2$ denote the regular h -neighborhood of the orthogonal projection $\Pi(\partial \Gamma)$ of $\partial \Gamma$ to \mathbb{R}^2 . There exist $C_g, \beta_s > 0$ so that if $\beta \leq \beta_s$ and F is a component of*

$$\mathbb{R}^2 - T_h(\Pi(\partial \Gamma)),$$

then each component of $\Pi^{-1}(F) \cap \Gamma$ is a graph over F of a function u with

$$|\nabla_{\mathbb{R}^2} u| \leq C_g \beta.$$

The second result is a scaled version of Theorem II.0.21 in [6] that gives conditions under which an embedded stable minimal disk contains a large multi-valued graph.

Theorem A.2 (Theorem II.0.21 in [6]). *Given $\tau > 0$, there exist $N_1, \Omega_1 > 0$ and $\varepsilon_1 > 0$ such that the following holds.*

Given $\delta \in (0, 1)$, let $\Sigma \subset \mathbb{B}(R_0)$ be a stable embedded minimal disk with $\partial\Sigma \subset \mathbb{B}(\delta r_0) \cup \partial\mathbb{B}(R_0) \cup \{x_1 = 0\}$ where $\partial\Sigma - \partial\mathbb{B}(R_0)$ is connected. Suppose the following hold:

1. $\Omega_1 r_0 < 1 < \frac{R_0}{\delta\Omega_1}$.
2. Σ contains an N_1 -valued graph Σ_g over $A(\delta, \delta r_0)$ with norm of the gradient is less than or equal to ε_1 .
3. $\Pi^{-1}(\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leq (\delta r_0)^2\}) \cap \Sigma_g^M \subset \{|x_3| \leq \varepsilon_1 \delta r_0\}$; here Σ_g^M denotes the middle sheet of Σ_g .
4. An arc $\tilde{\eta}$ in Σ connects Σ_g to $\partial\Sigma - \partial\mathbb{B}(R_0)$, where $\tilde{\eta} \subset \Pi^{-1}(D_{\delta r_0}) \cap [\Sigma - \partial\mathbb{B}(R_0)]$.

Then Σ contains a 10-valued graph Σ_d over $A(R_0/\Omega_1, \delta r_0)$ with norm of the gradient less than or equal to τ .

Remark A.3. Theorem A.2 is obtained by applying Theorem II.0.21 in [6] to $\frac{1}{\delta}\Sigma$. While in the statement of Theorem II.0.21 in [6], Σ_g is said to contain a 2-valued graph, the result above where Σ_g is said to contain a 10-valued graph also holds.

William H. Meeks, III at profmeeks@gmail.com

Mathematics Department, University of Massachusetts, Amherst, MA 01003

Giuseppe Tinaglia at giuseppe.tinaglia@kcl.ac.uk

Department of Mathematics, King's College London, London, WC2R 2LS, U.K.

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